Internals-I
GE06: Ordinary Differential Equations

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Total Marks: 32 **Time:** 90 minutes

Instructions: Attempt all questions. Show all working; Justifications for steps should come from theorems.

1. (8 marks) Answer **either** Part (a) or Part (b). Indicate your choice clearly at the top of your answer.

   (a) **Essay.**

   Second-order linear differential equations have been employed in macroeconomic models to describe business cycle fluctuations. Using the papers provided in class as your primary sources, write a critical essay ($\approx$ 400–500 words) in which you **either** defend **or** critique the use of second-order ODEs as a modelling tool for business cycles. Your essay must:

   (i) Briefly outline the mathematical setup of at least one model from the assigned papers;

   (ii) Identify at least **two** specific strengths **or** limitations (mathematical and/or economic) of the approach;

   (iii) State and defend a clear, argued position.

   If you cite any source beyond the assigned papers, write the full reference (author, title, journal/book, year) alongside the relevant claim. Uncited external claims will not receive credit.

   (b) **Derivation**

   Let $y_1(x)$ and $y_2(x)$ be two solutions of

   $$y'' + P(x)\,y' + Q(x)\,y = 0 \tag{1}$$

   on $[a, b]$, where P and Q are continuous. Their Wronskian is defined as

   $$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

   Prove that y_1 and y_2 are **linearly dependent** on $[a, b]$ **if and only if** $W(y_1, y_2) \equiv 0$ on $[a, b]$.

   Your proof must address both directions of the "if and only if" clearly and separately. You may assume that the Wronskian either vanishes nowhere or vanishes everywhere on $[a, b]$. Make sure to clearly specify the use of Picard's theorem.

2. (3 marks) If y_1, y_2 are the solutions to the differential equation

   $$y'' + yx + y\sin x + y\sqrt{4 - x^2} = 0,$$

   on $[0, 1]$, show that the Wronskian is constant. You may use any theorem/result proved in class, but quote it properly.

3. (8 marks) Solve any two of the following problems:

   (a) (4 marks) Solve the differential equation

$$(y + x^2)\,dx - x\,dy = 0.$$

   (b) (4 marks) Find the general solution of

$$y'' + 4y = 2x.$$

   (c) (4 marks) Given that $y_1 = e^x$ is a solution of

$$x\,y'' - (1 + x)\,y' + y = 0,$$

   find a second, linearly independent solution y_2.

4. (5 marks) Chebyshev's equation is

$$(1 - x^2)\,y'' - x\,y' + 25\,y = 0.$$

   Find an explicit solution $P(x)$ by assuming $y = \sum_{n=0}^{\infty} a_n x^n$ and $P(1) = 1$, deriving the recurrence relation for the coefficients. What is the region of convergence of $P(x)$?

5. Consider the pair of differential equations

$$y'' + P(x)\,y' + Q(x)\,y = 0 \qquad \text{and} \qquad y'' + Q(x)\,y' + P(x)\,y = 0,$$

   where P and Q are continuous on an interval $I = [0, 1]$.

   (a) (3 marks) Prove that if $f(x)$ is a common solution of both equations and $P(x) \neq Q(x)$ for all $x \in I$, then $f(x) = Ce^x$ for some constant C.

   (b) (5 marks) If $P(x) = x, Q(x) = -x + 1$, then find all common solutions to the pair of equations, with justification.