MADRAS SCHOOL OF ECONOMICS

UNDERGRADUATE PROGRAMME IN ECONOMICS (HONOURS) [2024-27]

SEMESTER FOUR [JANUARY – APRIL, 2026]

REGULAR EXAMINATION, APRIL-MAY 2026

Course Name: Differential Equations **Course Code:** GE06

Duration: 2 Hours **Maximum Marks: 60**

Instructions: *For part A write short answers (most preferably LESS than a couple of sentences). For part B, writing relevant formulae and quoting correct definitions helps me give you part marks.*

Part A: Answer all questions without justification (1 mark x 10 questions = 10 marks)

1. Write an example of a nonlinear first order differential equation in one variable.
2. Define the critical point of a nonlinear system of differential equation in two variables.
3. If
$$y(t) = t^2$$
then write down a constant coefficient second order differential equation whose solution is $y(t)$.
4. What is the Laplace transform of $e^{-t}u(t)$ (here $u(t)$ is the step function)?
5. Define an exact differential.
6. Draw a phase portrait of an unstable spiral.
7. Calculate all the Fourier series coefficients of $\cos x$. Write down values of a_n, b_n explicitly.
8. Define asymptotically stable critical point assuming that stable definition is known.
9. Write the formula for radius of convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$.
10. Write down a second order, constant coefficient linear differential equation whose Wronskian is constant.

Part B: Answer any five. (10 mark x 5 questions = 50 marks)

11. Prove that a homogeneous linear system has a node at $(0, 0)$ if the eigenvalues are both negative from first principles.

12. Solve the following differential equations and write down the general solution:

    (a) [5 marks] $(2y - x^3)dx = xdy$

    (b) [5 marks] $y'' + 2y' + y = xe^{-x}$ without using Laplace transform or power series methods.

13. Consider the differential equation
$$y'' + xy = 0.$$
Assume a solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$. Derive the recurrence relation. Work out the first two nonzero terms. Prove that the differential equation admits no nonzero polynomial solution.

14. Consider the initial value problem
$$\dot{x} = |x|^{1/2}, \quad x(0) = 0.$$

    (a) [7 marks] Construct infinitely many distinct solutions to the initial value problem, and verify that they satisfy the differential equation.

    (b) [3 marks] Explain why this does not contradict the Picard–Lindelöf existence and uniqueness theorem.

15. Answer the following questions on Laplace transform $\mathcal{L}$:

    (a) [4 marks] Show that
$$\mathcal{L}\{-te^{-t}\sin t + e^{-t}\sin t\} = \frac{s^2}{(s^2 + 2s + 2)^2}.$$

    (b) [6 marks] Using (a), solve the initial value problem
$$y'' + 2y' + 2y = -2e^{-t}\cos t, \quad y(0) = 0, \quad y'(0) = -1.$$

16. Solve the partial differential equation:
$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial t}, \quad 0 < x < \pi, \ t > 0,$$
$$u(0, t) = 0, \quad u(\pi, t) = 0, \ t > 0,$$
$$u(x, 0) = \sin x + \sin(3x), \quad 0 < x < \pi.$$

17. Let a, b be real numbers. For the DE system
$$x' = ax + y + bxy, y' = x + ay - bxy$$

    (a) [3 marks] Show that $(0, 0)$ is a hyperbolic critical point when $a \neq \pm 1$ and find the linearisation at $(0, 0)$.

    (b) [5 marks] Find the range of the values of a for which the critical point at $(0, 0)$ will be:

        i. a source node (all paths are approaching the node as $t \to \infty$).

        ii. a sink node (all paths are approaching the node as $t \to -\infty$)

        iii. a saddle.

    (c) [2 marks] Choose a convenient value for a for the saddle above, solve, and sketch the trajectories in the vicinity of the critical point, showing the direction of increasing t.