MADRAS SCHOOL OF ECONOMICS

POSTGRADUATE PROGRAMME IN DATA SCIENCE [2025-27]

SEMESTER 1 [JULY – NOVEMBER, 2025]

SUPPLEMENTARY EXAMINATION, JANUARY 2026

Course Name: Linear Algebra, **Course Code:** 115

Duration: 2 Hours **Maximum Marks: 60**

Instructions: *Any numerical computation also requires justification using theorems.*

Answer any six.

1. [10 marks] Define the problem that Recommendation Systems (used by Amazon, Spotify, Netflix, etc) are trying to solve. Set it up mathematically and explain clearly how SVD is used in recommendation systems. It will help to write the algorithm but you dont have to work out any numericals or present a program.

2. [10 marks] Show that every basis has the same number of elements in a finite dimensional vector space. You need to prove the Stienitz exchange too.

3. Answer the following questions:

   (a) [5 marks] State Cayley Hamilton theorem and use Cayley-Hamilton theorem to compute the inverse of
   $$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

   (b) [5 marks] Show that symmetric idempotent matrices project the domain onto the column space using spectral theorem.

4. Let
   $$A = \begin{pmatrix} 2 & 2 \\ 2 & 9 \end{pmatrix}$$
   Answer the following questions:

   (a) [3 marks] Perform LU decomposition of A.

   (b) [4 marks] Perform Schur's triangularisation of A.

   (c) [3 marks] Perform QR decomposition of A.

5. Answer the following questions:

   (a) [3 marks] Prove or disprove: Sum of squares of singular values of matrix A is the trace of the matrix AA^T.

   (b) [4 marks] Write the definition of an alternating form. Let $\omega : V^k \to \mathbb{R}$ be an alternating k-linear form. Prove that if $v_1, \ldots, v_k \in V$ are linearly dependent, then
   $$\omega(v_1, \ldots, v_k) = 0.$$

   (c) [3 marks] Construct a 2×2 matrix that is orthogonal and symmetric with real non-zero entries.

6. Show that a matrix is an orthogonal projection matrix iff it is symmetric idempotent.

7. Answer the following questions:

   (a) [5 marks] Construct the 3×3 matrix P that swaps coordinates, i.e. (x, y, z) goes to (x, z, y). What are it's eigenvalues and eigenvectors?

   (b) [5 marks] Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

   $$S(x, y, z) = (2x - y + z,\ x + y - z).$$

      (a) Find the matrix of S with respect to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$.

      (b) Compute $\text{rank}(S)$ and exhibit an explicit basis for $\text{range}(S)$.

8. Answer the following questions:

   (a) [3 marks] Show that if A and B are similar matrices, then $\det(A) = \det(B)$. Use this to prove that the determinant of a linear transformation is basis independent.

   (b) [5 marks] Consider the vector V space of all 2×2 matrices and let $S : V \to V$ be defined as $S(A) = A + A^T$. Show S is linear and work out the eigenvalue and eigenvectors of S.

   (c) [2 marks] Construct a matrix that is upper triangular and normal.