MADRAS SCHOOL OF ECONOMICS

POSTGRADUATE PROGRAMME IN DATA SCIENCE [2025-27]

SEMESTER 1 [JULY – NOVEMBER, 2025]

REGULAR END TERM EXAMINATION, DECEMBER 2025

Course Name: Linear Algebra, **Course Code:** 115

Duration: 2 Hours **Maximum Marks: 60**

Instructions:

Answer any six.

1. [10 marks] Define the problem that Recommendation Systems (used by Amazon, Spotify, Netflix, etc) are trying to solve. Set it up mathematically and explain clearly how SVD is used in recommendation systems. It will help to write the algorithm but you dont have to work out any numericals or present a program.

2. [10 marks] Let V, W be vector spaces over the real numbers. Let $T : V \to W$ be a linear transformation, then $rank(T) + nullity(T) = \dim(V)$.

3. Answer the following questions:

   (a) [3 marks] If A, B have a common orthogonal eigenbasis, prove that $AB = BA$.

   (b) [7 marks] Suppose that we are given an m-vector b and an $m \times n$ matrix A. Find x such that $||b - Ax||^2$ is minimum. You may assume the SVD of A but nothing else.

4. Let

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 7 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

   Answer the following questions:

   (a) [3 marks] Perform LU decomposition of A.

   (b) [4 marks] Perform Schur's triangularisation of A.

   (c) [3 marks] Perform QR decomposition of A.

5. Answer the following questions:

   (a) [2 marks] Prove or disprove: The product of two symmetric matrices is symmetric.

   (b) [4 marks] Construct the 2×2 matrix P that orthogonally projects any point (x, y) onto the line $y = 4x$. Show that $I - 2P$ reflects a point about that line.

   (c) [4 marks] Construct an example of a 3×3 matrix A with real entries whose characteristic polynomial is $p_A(t) = t^3 + t + 1$.

6. Answer the following questions:

   (a) [3 marks] Prove that product of two lower triangular matrices is lower triangular.

   (b) [4 marks] Derive the formula for the 3×3 determinant using the axioms of a multilinear alternating form.

   (c) [3 marks] Construct an example of a 3×3 matrix that positive definite but not symmetric.

7. Answer the following questions:

(a) [5 marks] Construct the 3×3 matrix P that cycles three coordinates, i.e. (x, y, z) goes to (z, x, y). What are it's eigen values and eigenvectors?

(b) [5 marks] Consider the vector V space of all 2×2 matrices and let $S : V \to V$ be defined as $S(A) = A^T$. Show S is linear and work out the eigenvalue and eigenvectors of S.

8. Answer the following questions:

   (a) [3 marks] Show that if A and B are similar matrices, then $trace(A) = trace(B)$. Use this to prove that the trace of a linear transformation is basis independent.

   (b) [5 marks] Construct a 4×4 matrix A with entries from the set $\{\pm 1\}$ which has a 2-dimensional invariant subspace. Block triangularise the matrix.

   (c) [2 marks] Prove $\det(AB) = \det(A)\det(B)$ using the alternating form definition of determinant.