Linear Transformations, Matrices, and Rank

Srikanth Pai
Assistant Professor, MSE, Chennai.

September 14 2025

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by
$$T(x, y, z) = (x + y,\ y + z,\ x + z).$$
   (a) Verify that T is a linear transformation.
   (b) Determine the range of T and describe it as a subspace of $\mathbb{R}^3$.
   (c) Compute the rank of T.

2. Consider $S : \mathbb{R}^3 \to \mathbb{R}^2$ given by
$$S(x, y, z) = (2x - y + z,\ x + y - z).$$
   (a) Find the matrix representing S with respect to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$.
   (b) Compute the rank of S and exhibit a basis for its range.

3. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by
$$T(x, y) = (x + 2y,\ 3x + y).$$
   (a) Write down the matrix of T with respect to the standard basis.
   (b) Let $B = \{(1, 1),\ (1, -1)\}$. Compute the matrix of T with respect to the basis B.
   (c) Explain how the change of basis affects the representation of the transformation but not the transformation itself.

4. For each of the following matrices, compute the column rank and exhibit a basis for the column space:
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by
$$T(x, y, z) = (x - y,\ y - z,\ z - x).$$
Compute the matrix of T in the standard basis. Determine the set of all possible outputs of T and state its dimension.

6. Consider the matrix

$$M = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 1 & 3 & 3 \end{bmatrix}.$$

Find the rank of M. Exhibit two different bases for the column space of M.

7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection across the line $y = x$.

   (a) Write down the matrix of T in the standard basis.

   (b) Identify two independent vectors that remain unchanged (up to sign) under this transformation.

8. Suppose V is the space of all 2×2 real matrices. Define

$$T : V \to V, \quad T(A) = A^T.$$

Describe the set of all possible outputs of T (that is, all matrices of the form A^T). What is the dimension of this set?

9. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(x, y) = (x + y,\ x - y).$$

Is T onto? Justify your answer by showing whether every vector in $\mathbb{R}^2$ can be written as $T(x, y)$ for some (x, y).

10. Let P_2 denote the vector space of real polynomials of degree at most 2. Define

$$D : P_2 \to P_1, \quad D(p) = p',$$

where p' is the derivative of p.

   (a) Write down D applied to the basis $\{1, x, x^2\}$ of P_2.

   (b) Determine the set of all possible outputs of D.

   (c) Is D onto P_1? Explain.

11. Let $C[0, 1]$ denote the vector space of continuous real functions on $[0, 1]$. Define

$$T : C[0, 1] \to \mathbb{R}, \quad T(f) = \int_0^1 f(x)\, dx.$$

Show that T is a linear transformation. Is T onto $\mathbb{R}$? Justify.