Internals 2: Linear Algebra

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Maximum Marks: 20 **Time: 90 minutes**

Instructions: Show all working clearly. Partial credit will be awarded for correct reasoning even if the final answer is incomplete.

Part A (3 marks $\times$ 4 = 12 marks)

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1. Compute the characteristic polynomial of
$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$
Compute eigenspaces for each eigenvalue. Is A diagonalizable?

2. For
$$B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix},$$
find the minimal B-invariant vector subspace of $\mathbb{R}^3$ that contains $(1, 1, 1)$.

3. Let
$$A = \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}.$$
Use the Gram–Schmidt process to obtain orthogonal Q and upper-triangular R such that $A = QR$.

4. Prove that the trace of a square matrix equals the sum of its eigenvalues.

Part B (5 marks $\times$ 2 = 10 marks)

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5. Consider the matrix
$$A = \begin{pmatrix} 8 & 3 & 3 & -3 & -6 \\ 3 & 8 & 3 & -3 & -6 \\ 3 & 3 & 8 & -3 & -6 \\ 3 & 3 & 3 & 2 & -6 \\ 3 & 3 & 3 & -3 & -1 \end{pmatrix}.$$
   (a) Write A in the form $A = aI + b\,xy^T$ where $x = (1, 1, 1, 1, 1)^T$ and $y \in \mathbb{R}^5$. Find a, b, and y.

(b) Compute $\det(A)$.

(c) Compute A^{-1}.

6. Define

$$\langle x, y\rangle_{new} = x^{\mathsf{H}} \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix} y, \quad x, y \in \mathbb{C}^2.$$

(a) The adjoint of T with respect to $\langle \cdot, \cdot \rangle_{new}$ is a transformation S such that $\langle Tx, y\rangle_{new} = \langle x, Sy\rangle_{new}$ for all x, y. Let

$$T = \begin{pmatrix} 1 & 1+i \\ 1 & -1 \end{pmatrix}.$$

Find an adjoint of T. *(Warning: Adjoint is not Adj.)*

(b) Find the orthogonal projection matrix P (with respect to this inner product) onto the line spanned by $v = (1, -1)$. Verify that $P^2 = P$ and adjoint of P is itself. Is $P^H = P$?