Linear Independence & Dependence

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Homework Assignment

Instructions: Show all work clearly. For computational problems, verify your answers. For proof problems, write complete mathematical arguments.

1. Determine whether the following sets of vectors in $\mathbb{R}^3$ are linearly independent or dependent:
   (a) $\{(1,2,3),(2,1,0),(0,3,6)\}$
   (b) $\{(1,0,1),(0,1,1),(1,1,0)\}$
   (c) $\{(2,-1,3),(1,0,1),(0,-1,1)\}$
2. In $\mathbb{R}^4$, determine whether the set
   $$\{(1,1,0,0),(1,0,1,0),(0,1,1,0),(0,0,1,1)\}$$
   is linearly independent.
3. Find all values of k for which the vectors $(1,2,k)$, $(2,k,1)$, and $(k,1,2)$ are linearly dependent in $\mathbb{R}^3$.
4. Consider the vector space $M_{2\times 2}(\mathbb{R})$ of all 2×2 real matrices. Determine whether the following set is linearly independent:
   $$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$
5. In the space of 2×2 symmetric matrices, show that
   $$S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
   form a linearly independent set, and explain why no fourth 2×2 symmetric matrix can be added to maintain linear independence.

   Hint: Think about the dimension of the space of 2×2 *symmetric matrices.*

6. In the vector space $P_3(\mathbb{R})$ of polynomials of degree at most 3, determine whether the set

$$\{1,\ x+1,\ x^2-1,\ x^3+x\}$$

is linearly independent.

7. Consider $p_1(x) = x^2 + ax + 1$, $p_2(x) = x^2 + x + a$, and $p_3(x) = ax^2 + x + 1$ in $P_2(\mathbb{R})$.

   (a) For which values of a are these polynomials linearly dependent?

   (b) When $a = 2$, express one of the polynomials as a linear combination of the other two.

8. Let $p_1(x) = x^2+1$, $p_2(x) = x^2+x$, and $p_3(x) = x+1$. Consider the linear combination

$$c_1p_1(x) + c_2p_2(x) + c_3p_3(x) = 0 \quad \text{for all } x.$$

   (a) Find all solutions (c_1, c_2, c_3).

   (b) What can you conclude about the linear independence of $\{p_1, p_2, p_3\}$?

   (c) Now suppose we only require $c_1p_1(0)+c_2p_2(0)+c_3p_3(0) = 0$ and $c_1p_1(1)+c_2p_2(1)+c_3p_3(1) = 0$. How many solutions are there now? What does this tell you about evaluating polynomials at specific points?

9. Consider the vector space of all real-valued functions on $\mathbb{R}$. Determine whether the following sets are linearly independent:

   (a) $\{e^x, e^{2x}, e^{3x}\}$

   (b) $\{\sin x, \cos x, \sin(x + \pi/4)\}$

   (c) $\{1, x, x^2, e^x\}$

   Hint: For part (b), use the angle addition formula.

10. Let $f_1(x) = x,\ f_2(x) = x^2,\ f_3(x) = x^3$.

    (a) Show that $\{f_1, f_2, f_3\}$ is linearly independent in the vector space of all functions $\mathbb{R} \to \mathbb{R}$.

    (b) Restrict these functions to the domain $\{0, 1, 2\}$. Are they still linearly independent?

    (c) What about when restricted to the domain $\{0, 1\}$?

11. Let V be a vector space and $\{v_1, v_2, v_3\}$ linearly independent.

    (a) Prove that $\{v_1,\ v_1 + v_2,\ v_1 + v_2 + v_3\}$ is also linearly independent.

    (b) More generally, prove that $\{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$ is linearly independent iff $\{v_1, v_2, v_3\}$ is.

12. Let V be a vector space over a field F, and $S = \{v_1, \ldots, v_n\}$.

(a) Prove that S is linearly dependent iff at least one vector in S can be written as a linear combination of the others.

(b) If S is linearly dependent, prove there exists a proper subset $T \subset S$ with $\text{span}(T) = \text{span}(S)$.

13. Let $u, v, w \in V$.

    (a) If $\{u, v\}$ and $\{u, w\}$ are linearly independent, must $\{u, v, w\}$ be independent? Prove or counterexample.

    (b) If $\{u, v\}, \{v, w\}, \{u, w\}$ are all linearly independent, must $\{u, v, w\}$ be independent? Prove or counterexample.

14. **The Vandermonde Connection.** Consider in $\mathbb{R}^n$:

$$v_i = (1, a_i, a_i^2, \ldots, a_i^{n-1}), \quad i = 1, \ldots, n$$

with a_i distinct reals.

    (a) For $n = 3$, show v_1, v_2, v_3 are linearly independent.

    (b) What happens if two a_i coincide?

15. **Linear Independence Across Spaces.** Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}.$$

    (a) Are A, B, C linearly independent in $M_{2\times 2}(\mathbb{R})$?

    (b) As linear maps $T_A(x) = Ax$, etc., are they linearly independent in $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$?

    (c) Relate your answers in (a) and (b).

Reminder: Linear independence means the only solution to $c_1 v_1 + \cdots + c_n v_n = 0$ is $c_1 = \cdots = c_n = 0$.