Problem Set: Eigenvalues, Diagonalisation, and Invariant Subspaces

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1. Find the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

   Verify that the trace and determinant match the sum and product of its eigenvalues.

2. Compute all eigenvalues and eigenvectors of the matrix

$$B = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}.$$

   Check whether B is diagonalizable.

3. Two linear transformations $T_1, T_2 : \mathbb{R}^n \to \mathbb{R}^n$ have the same characteristic polynomial.
   (a) Is it necessarily true that $T_1 = T_2$?
   (b) Is it necessarily true that T_1 and T_2 have the same set of eigenvalues?
   (c) Suppose both T_1 and T_2 have a full set of linearly independent eigenvectors. Must they then be the same transformation?
   Give a proof or a counterexample.

4. Prove that for any 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\operatorname{trace}(M^2) = (\operatorname{trace}(M))^2 - 2\det(M).$$

   Then verify this identity for $M = \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$.

5. Two n x n matrices A,B are *similar* if there is an invertible matrix S so that $A = S^{-1}BS$. Show that A and B are similar is equivalent to saying that there is a basis $\mathcal{B}$ such that B is the matrix of the linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$.

6. Let $V = \mathbb{R}^4$ and T be the linear operator with matrix

$$T = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

   Find the characteristic polynomial, and eigenvectors of T. Is T diagonalizable? Identify all T-invariant subspaces.

7. Let A_n denote the $n \times n$ matrix with 1's on the superdiagonal and zeros elsewhere. Compute its characteristic polynomials. What are its eigenvalues? For which n is A_n diagonalizable? Find the invariant subspaces.

8. Let $T : V \to V$ be linear and $v \in V$. Define

$$W_{v,T} := \operatorname{span}\{v, Tv, \ldots, T^k v\},$$

where k is the smallest integer for which $T^{k+1}v$ is a linear combination of $\{v, Tv, \ldots, T^k v\}$.

   (a) Prove that $W_{v,T}$ is T-invariant.

   (b) Prove that $W_{v,T}$ is the minimal T-invariant subspace containing v, i.e. prove that any T-invariant subspace that contains v will automatically contain $W_{v,T}$.

   (c) Show there is a unique monic polynomial p of least degree with $p(T)v = 0$, and that $W_{v,T} = \ker p(T)$.

   Concrete 3×3 *instance.* Let $V = \mathbb{R}^3$ and

$$T = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \qquad v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

   (i) Compute v, Tv, T^2v, T^3v explicitly.

   (ii) Determine the smallest k such that $T^{k+1}v$ is a linear combination of $\{v, Tv, \ldots, T^k v\}$. Hence write the linear relation among $v, Tv, \ldots, T^k v$.

   (iii) Find the monic polynomial p of least degree with $p(T)v = 0$.

   (iv) Compute a basis for $\ker p(T)$ and verify that $\ker p(T) = \operatorname{span}\{v, Tv, \ldots, T^k v\}$.

   (v) Decide whether $\{v, Tv, \ldots, T^k v\}$ is linearly independent. Justify your answer from your computations above.

9. Show that if A and B are *similar* matrices, then $\operatorname{trace}(A) = \operatorname{trace}(B)$. Use this to prove that the trace of a linear transformation is basis independent.

10. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Verify Schur's theorem explicitly by finding a unitary matrix U such that U^*AU is upper triangular.

11. Consider

$$A = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 0 & 3 & 0 & 1 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & 0 & 3 \end{pmatrix}.$$

   (a) Find a nontrivial 2-dimensional A-invariant subspace without guessing coordinate subspaces. (Hint: look at vectors with $x_1 = x_3$ and $x_2 = x_4$.)

   (b) Choose a basis adapted to your invariant subspace and write the matrix of A in this basis. Show it is block upper triangular and identify the 2×2 diagonal blocks explicitly.

   (c) Deduce the eigenvalues of A from your block form. Decide whether A is diagonalizable, justifying your answer from your blocks.

12. Let $A, B \in M_n(\mathbb{R})$ be commuting diagonalizable matrices. Prove that A and B are simultaneously diagonalizable. Then construct two commuting 4×4 matrices that are not simultaneously diagonalizable, and explain precisely why the obstruction arises.