Problem Set: Determinants and Adjugates

Srikanth Pai, Asst. Professor, MSE

1. Let $f : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the unique alternating trilinear form with $f(e_1, e_2, e_3) = 1$. Compute
$$f\big((1,1,0),(1,0,1),(2,3,1)\big).$$

2. Let A be any $n \times n$ matrix. Suppose one performs an elementary row operation of the form
$$R_i \leftarrow R_i + kR_j \quad (i \neq j).$$
Explain, using the multilinear property of the determinant, why this operation does not change $\det(A)$.

3. Compute the determinants of the following matrices:
$$A_1 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

4. Show that the determinant of a triangular matrix is the product of the diagonal entries.

5. Compute the following determinants using only basic determinant properties (row and column operations, multilinearity, or induction):
$$D_1 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}, \qquad D_2 = \begin{vmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{vmatrix},$$
$$D_3 = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 5 & 0 \\ 0 & 6 & 7 & 8 \\ 0 & 0 & 9 & 10 \end{vmatrix}, \qquad D_4 = \begin{vmatrix} x & 1 & 0 & 0 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 0 & 0 & 1 & x \end{vmatrix}.$$

6. Let $x, y \in \mathbb{R}^n$ be column vectors. Show that
$$\det(I_n + xy^T) = 1 + y^T x.$$

7. Let A be an invertible $n \times n$ matrix. Prove that

$$\det(\operatorname{adj}(A)) = (\det A)^{n-1}.$$

8. Consider the $n \times n$ Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}.$$

Compute $\det(V)$ and determine whether $\operatorname{adj}(V)$ is also a Vandermonde-type matrix.

9. Show that if P is a projection matrix (i.e. $P^2 = P$), then $\det(P)$ equals either 0 or 1. What can you say about $\det(I - P)$?

10. Prove or disprove: for all $n \times n$ matrices A, B,

$$\det(A + B) = \det(A) + \det(B).$$

If false, provide a specific counterexample and explain conceptually why the property fails in general.

11. Prove or disprove: if P is any real square matrix with $\det(P) < 1$, then $I - P$ must be invertible.

12. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map

$$T(x, y, z) = (x + y,\ y + z,\ x + z).$$

Compute $\det(T)$ and interpret geometrically the transformation of volume and orientation.

13. (Bonus) Let $(r, \theta_1, \theta_2, \ldots, \theta_{n-1})$ be the spherical coordinate system in $\mathbb{R}^n$ defined by

$$\begin{aligned} x_1 &= r\cos\theta_1, \\ x_2 &= r\sin\theta_1\cos\theta_2, \\ x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3, \\ &\ \vdots \\ x_{n-1} &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1}, \\ x_n &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1}. \end{aligned}$$

(a) Compute the Jacobian determinant

$$J(r, \theta_1, \ldots, \theta_{n-1}) = \det\left(\frac{\partial(x_1, \ldots, x_n)}{\partial(r, \theta_1, \ldots, \theta_{n-1})}\right).$$

Show that

$$|J| = r^{\,n-1}\,(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots(\sin\theta_{n-2}).$$

(b) Hence derive a recursive formula for the volume V_n of the n-dimensional unit ball in terms of V_{n-2}.

(c) Compute V_n explicitly and verify that

$$V_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}.$$