Problem Set: Inner Product Spaces and Alternating Forms

1. Let $\langle (x_1, x_2), (y_1, y_2) \rangle = 2x_1y_1 + 2x_2y_2 + x_1y_2 + x_2y_1$ on $\mathbb{R}^2$.
   (a) Determine whether this defines an inner product.
   (b) If it does, find an orthonormal basis for $\mathbb{R}^2$ with respect to this inner product.

2. In $\mathbb{R}^4$, consider the basis
$$v_1 = (1, 1, 0, 0), \quad v_2 = (1, 0, 1, 0), \quad v_3 = (0, 1, 1, 1), \quad v_4 = (1, 1, 1, 1).$$
   (a) Apply the Gram–Schmidt process to obtain an orthogonal basis $\{u_1, u_2, u_3, u_4\}$.
   (b) Express each v_i as a linear combination of $\{u_1, u_2, u_3, u_4\}$.

3. Let
$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$
   Find an orthogonal matrix Q and an upper triangular matrix R such that their product equals A, i.e. $A = QR$.

4. Consider a 3×3 matrix
$$Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ a & b & c \end{pmatrix}.$$
   Determine (a, b, c) so that Q is an orthogonal matrix. How many distinct such rows exist?

5. Construct the 2×2 matrix that orthogonally projects any point (x, y) onto the line $y = mx$ in $\mathbb{R}^2$.

6. Prove that if P is an orthogonal projection matrix, then $I - 2P$ is an orthogonal matrix.

7. Let x be a unit vector in $\mathbb{R}^d$. Define $P(y) = (y^T x)x$.

(a) Show that P is an orthogonal projection operator.

(b) Identify the subspace onto which P projects and find $\operatorname{rank}(P)$.

(c) Compute $I - 2P$ explicitly for $\mathbb{R}^2$ and interpret the transformation geometrically.

8. Let P and Q be orthogonal projection matrices on $\mathbb{R}^n$. Prove that $PQ = QP$ if and only if the subspaces $\operatorname{Im}(P)$ and $\operatorname{Im}(Q)$ are mutually orthogonal or one is contained in the other.

9. Let $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f((x_1, x_2), (y_1, y_2)) = \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}.$$

(a) Show that f is an alternating bilinear form.

(b) Determine the dimension of the space of all alternating bilinear forms on $\mathbb{R}^2$.

10. Let $f : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be an alternating trilinear form with $f(e_1, e_2, e_3) = 1$. Compute

$$f\big((1,1,1), (1,1,0), (2,3,1)\big).$$