HW6: Linear Algebra

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Instructions. Answer all questions. Where requested, give brief justifications for each step.

1. State the Rank–Nullity Theorem for a linear map $T : V \to W$ between finite-dimensional vector spaces. Define the terms rank and nullity. Explain briefly why both sides of the identity have to be finite.

2. Let
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & -1 \end{pmatrix}.$$
Compute $\operatorname{rank}(A)$ and $\dim \ker(A)$. Exhibit a basis for the column space and a basis for the null space.

3. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be given by
$$T(x_1, x_2, x_3, x_4) = (x_1 + 2x_2 - x_3,\ x_2 + x_3 + 3x_4,\ 2x_1 - x_2 + 4x_4).$$
   (a) Write the matrix $[T]$ relative to the standard bases.
   (b) Compute $\operatorname{rank}(T)$ and $\dim \ker(T)$ using row reduction.
   (c) Give an explicit basis for $\ker(T)$.

4. Let V be a finite-dimensional vector space and $S, T : V \to V$ be linear maps such that $\operatorname{im}(S) \subseteq \ker(T)$. Show that
$$\operatorname{rank}(S) + \operatorname{rank}(T) \leq \dim V.$$
Give an example (on $\mathbb{R}^3$) where the inequality is strict.

5. Prove that for linear maps $S : U \to V$ and $T : V \to W$ between finite-dimensional spaces,
$$\operatorname{rank}(T \circ S) \leq \min\{\operatorname{rank}(S), \operatorname{rank}(T)\}.$$
Then give a condition when equality can hold.

6. Let $T : V \to W$ be linear with $\dim V = \dim W < \infty$. Prove that T is injective iff it is surjective. Deduce the Rank–Nullity statement for this case.

7. On $\mathbb{R}^2$ define $\langle x, y \rangle_Q = x^T Q y$ where $Q = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Verify that this is an inner product. Find the Q-orthogonal complement of the line spanned by $(1, -1)$. Give an example of a fake inner product on $\mathbb{R}^2$ that satisfies the first axiom of inner product, and fails the other axioms.

8. On the space $C[0,1]$ of continuous real functions, define

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$

   (a) Compute $\langle f, g\rangle$ when $f(x) = x$, $g(x) = 1 - x$.
   (b) Compute $\|f\| = \sqrt{\langle f, f\rangle}$ for $f(x) = x^2$.

9. On $\mathbb{R}^3$ define

$$\langle u, v\rangle = \alpha u_1 v_1 + \beta u_2 v_2 + \gamma u_3 v_3,$$

where $\alpha, \beta, \gamma > 0$. Show this is an inner product. Compute $\|u\|$ for $u = (1, 2, 3)$ when $(\alpha, \beta, \gamma) = (1, 2, 3)$.