Problem Set: Final Set (Schur, Spectral, SVD, Positive definite matrices)

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1. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Compute its eigenvalues and an orthonormal basis of eigenvectors. Verify explicitly that A is symmetric positive definite.

2. The Cayley-Hamilton theorem states that every matrix satisfies its own characteristic equation. Prove this statement for diagonalisable matrices.

3. Use Cayley-Hamilton theorem to compute the inverse of $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

4. Prove or disprove: Let A be a symmetric 3×3 real matrix. Show that if $x^\top Ax = 0$ for all $x \in \mathbb{R}^3$, then $A = 0$.

5. Let A be a normal matrix, then we know that
$$A = U\Sigma U^H.$$
Show that
$$A = \sum_{k=1}^{n} \sigma_i u_i u_i^H$$
where u_i is the ith column of U and $\sigma_i = \Sigma_{ii}$.

6. Solve the following problems:

   (a) Recall that for a matrix T, matrix multiplication formula gives
   $$[T^3]_{ij} = \sum_{l,m} T_{il} T_{lm} T_{mj}.$$
   Now suppose T is upper triangular 3×3 matrix with diagonal entries 0, show that $T^3 = 0$. [Hint: convince yourself that $[T^3]_{ij}$ is non-zero only if $i < l < m < j$ but there are only three indices.]

   (b) A nilpotent matrix N is a matrix such that $N^k = 0$ for some k. Suppose the nilpotent matrix N is 3×3 matrix. Show $N^3 = 0$. [Hint: Schur's triangularisation.]

7. If A, B have a common orthogonal eigenbasis, prove that $AB = BA$.

8. Show that symmetric idempotent matrices project the domain onto the column space. [Hint: Use spectral theorem.]

9. Let A be symmetric positive semidefinite. Prove that there exists a unique symmetric positive semidefinite matrix B with $B^2 = A$.

10. Compute the SVD and Schur's triangularisation of

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

$$B = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

11. Let

$$S^n = \{x \in \mathbb{R}^n \mid ||x||^2 = 1\}$$

denote the unit sphere in n-dimensional space. For $a \in \mathbb{R}^n$ with non-negative entries, a set of the form

$$E = \{x \in \mathbb{R}^n \mid a_1 x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2 = 1\}$$

is called an ellipsoid in $\mathbb{R}^n$. Given a $m \times n$ matrix, consider it as a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$. Show that A maps the unit sphere in $\mathbb{R}^n$ to an ellipsoid.

This is the geometric interpretation of a famous theorem.

12. The singular value decomposition is very useful in studying the linear least squares problem. This arises in maximum likelihood estimation of normally distributed vectors. Suppose that we are given an m-vector $\mathbf{b}$ and an $m \times n$ matrix A. Find $\mathbf{x}$ such that $\|\mathbf{b} - A\mathbf{x}\|_2$ is minimum. [You may assume the SVD of $A = U\Sigma V^T$ for the answer.]

13. Let $\langle x, y\rangle_A = x^T A y$ be an inner product. Show that there exists an invertible matrix T such that

$$\langle x, y\rangle_A = \langle Tx, Ty\rangle_2.$$

Give an explicit formula for T in terms of A.

14. Solve the following problems:

(a) Show that

$$tr(AA^T) = \sum_{i,j} a_{ij}^2.$$

(b) Is it possible for a transition probability matrix (tpm) of a 3-state time homogenous DTMC to have a singular value of $\sqrt{5}$? [You can replace tpm with any matrix with entries between 0 and 1.]

15. Using the SVD of A, compute the SVD of A^T and A^{-1}.