Basis and Dimension

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Linear Algebra for Data Science Year 1 students.

Problems

1. Let B be a subset of a vector space V. Prove that the following are equivalent:
   (a) B spans V and is linearly independent.
   (b) B is a minimal spanning set.
   (c) B is a maximal independent set.
   (d) B spans V and every vector of V has a unique representation in terms of B.

2. In $\mathbb{R}^4$, let

   $$A = \{(1,0,1,0),(0,1,1,1)\}, \quad B = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),(1,1,1,1)\}.$$

   Use the Steinitz Exchange Lemma to find $S \subseteq B$ such that $A \cup S$ spans $\mathbb{R}^4$. Show steps. Deduce that every spanning set of $\mathbb{R}^4$ has at least 4 vectors.

3. Use the Steinitz Exchange Lemma to prove that any two bases of a finite-dimensional vector space have the same cardinality.

4. In $\mathbb{R}^3$, let

   $$S = \{(1,0,0),(0,1,0),(0,0,1),(1,1,0)\}, \quad T = \{(1,1,1),(1,2,0)\}.$$

   Carry out the Steinitz exchange procedure explicitly and verify the result spans $\mathbb{R}^3$. Explain why, at each step, some original w_i must be exchanged out. Conclude that $n \leq m$.

5. (a) Show that any independent set in V can be extended to a basis.
   (b) Show that any spanning set in V contains a basis.

6. Find the dimension of each subspace below by exhibiting a basis:
   (a) $\{(x,y,z,w) \in \mathbb{R}^4 : x+y+z+w = 0\}$.
   (b) $\{p(x) \in \mathbb{R}[x] : p$ is a polynomial of degree at most 4, with $p(1) = 0\}$.
   (c) The space of 2×2 symmetric real matrices.

(d) The span of $\{(1,1,0,0),(0,1,1,0),(0,0,1,1),(1,0,0,1)\}$ in $\mathbb{R}^4$.

7. Prove: if $T : V \to W$ is linear and $B = \{v_1, \ldots, v_n\}$ is a basis of V, then the values $T(v_1), \ldots, T(v_n)$ determine T uniquely on all of V. Conclude that specifying the images of a basis is equivalent to specifying the linear map.

8. Compute the standard-basis matrix for each linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$. Show computations with respect to the standard basis.

   (a) Rotation by angle θ about the z-axis.

   (b) Rotation by angle θ about the x-axis.

   (c) Uniform expansion about the origin by factor 3.

   (d) Reflection through the origin.

   (e) Reflection in the xy-plane.

   (f) Reflection in the plane through the origin with normal vector $n = (1,1,0)$. Use the formula

   $$R = I - 2\frac{nn^T}{n^Tn}$$

   and compute the matrix explicitly.

9. Let $B = \{b_1, b_2, b_3\}$ be the standard basis of $\mathbb{R}^3$. For each linear transformation below, give the matrix with respect to the standard basis and compute the image of the vector indicated.

   (a) Define T by $T(e_1) = (1,0,1),\ T(e_2) = (0,2,0),\ T(e_3) = (1,1,1)$. Find the matrix of T and compute $T(2,1,-1)$.

   (b) Define S by $S(e_1) = (0,1,0),\ S(e_2) = (1,0,0),\ S(e_3) = (0,0,-1)$. Find the matrix of S and compute $S(3,-2,4)$.

10. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by $90°$, and let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the line $y = x$.

    (a) Find the matrices of T and S with respect to the standard basis.

    (b) Compute the matrices of TS and ST.

    (c) Identify each transformation geometrically.

11. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by $90°$.

    (a) Find the matrix of T in the standard basis.

    (b) Find the matrix of T in the basis $\{(1,1),(1,-1)\}$.

    (c) Show the two matrices are similar.

12. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(x,y,z) = (x+y,\ y+z,\ x+z)$.

(a) Find a basis for the image of T.

(b) Determine the dimension of the image.

(c) Is T surjective? Justify.

13. Let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be projection onto the xy-plane: $P(x, y, z) = (x, y, 0)$.

(a) Find the matrix of P in the standard basis.

(b) Verify $P^2 = P$.

14. Construct or prove impossible:

(a) Find a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(1, 0) = (2, 1)$ and $T(0, 1) = (1, -1)$. Give its matrix.

(b) Does there exist a linear map $S : \mathbb{R}^2 \to \mathbb{R}^2$ with $S(1, 0) = (1, 0)$, $S(0, 1) = (0, 1)$, and $S(1, 1) = (2, 3)$? Justify.