Homework #3 - Machine Learning

Srikanth Pai, Asst. Professor, MSE

January 24, 2026

Problem Set

1. Let $x, y \in \mathbb{R}^n$ and let $A \in \mathbb{R}^{n\times n}$ be a constant matrix.

   (a) Show that
   $$\nabla_x(x^\top y) = y.$$

   (b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and define
   $$S = \{x \in \mathbb{R}^n : f(x) \le 0\}.$$
   Show that S is a convex set.

   (c) Prove the convex hull of a subset of euclidean space (defined in class) is convex.

   (d) Let
   $$\Delta^n = \left\{ p \in \mathbb{R}^n : p_i \ge 0 \text{ for all } i,\ \sum_{i=1}^{n} p_i = 1 \right\}.$$
   Show that Δ^n is a convex set.

2. Show that
   $$\nabla_x(x^\top Ax) = (A + A^\top)x.$$

3. Deduce that if A is symmetric, then
   $$\nabla_x(x^\top Ax) = 2Ax.$$

4. Let $x \in \mathbb{R}^n$ and let $A \in \mathbb{R}^{n\times n}$ be a constant matrix.

   (a) Compute
   $$\nabla_x^2(x^\top Ax).$$

   (b) Show that if A is symmetric, then
   $$\nabla_x^2(x^\top Ax) = 2A.$$

5. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix and define
   $$f(x) = x^\top Ax.$$
   Show that f is convex on $\mathbb{R}^n$ if and only if A is positive semidefinite.

6. Let
   $$f(x) = \|Ax - b\|^2, \qquad A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m.$$

(a) Compute $\nabla f(x)$ and $\nabla^2 f(x)$.
(b) Find all critical points.
(c) Determine whether every critical point is a global minimiser.

7. Let
$$f(x) = \|Ax - b\|^2 + \lambda \|x\|^2, \qquad \lambda > 0.$$
   (a) Compute the gradient and Hessian.
   (b) Show that f is strictly convex.
   (c) Find the unique global minimiser.

8. Let
$$f(x, y) = e^{x+y} + x^2 + y^2.$$
   (a) Compute ∇f and $\nabla^2 f$.
   (b) Using the Hessian criterion, show that f is convex.
   (c) Find the global minimum.

9. Let
$$f(x) = \log(1 + e^x).$$
   (a) Compute $f'(x)$ and $f''(x)$.
   (b) Use the second derivative test to show that f is convex on $\mathbb{R}$.
   (c) Find all critical points.

10. Let
$$f(x) = x^4 - 4x^2.$$
   (a) Find all critical points.
   (b) Classify them using the second derivative test.
   (c) Identify which critical points are global minima.