Homework #2 - Machine Learning

Srikanth Pai, Asst. Professor, MSE

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Problem Set

1. Let $C_1 \subset \mathbb{R}^m$ and $C_2 \subset \mathbb{R}^n$ be convex sets. Prove or disprove that the Cartesian product

$$C_1 \times C_2 \subset \mathbb{R}^{m+n}$$

is convex.

2. Let $\{C_\alpha\}_{\alpha \in A}$ be a family of convex subsets of $\mathbb{R}^n$. Prove or disprove that

$$\bigcap_{\alpha \in A} C_\alpha$$

is convex.

3. Give an explicit example of two convex subsets of $\mathbb{R}^2$ whose union is not convex. Justify your answer.

4. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Prove that the unit ball

$$B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$$

is a convex set.

5. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Prove that the function

$$f(x) = \|x\|$$

is convex.

6. Define the ℓ_∞-norm on $\mathbb{R}^n$ by

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

Show that $\|\cdot\|_\infty$ satisfies the axioms of a norm.

7. You may assume Hölder's inequality. Use it to show that for any $x \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} |x_i| \leq n^{1-1/p} \, \|x\|_p \qquad (p \geq 1).$$

In particular, deduce the inequality

$$\|x\|_1 \leq \sqrt{n} \, \|x\|_2.$$

8. Let $A \in \mathbb{R}^{m \times n}$. The Moore–Penrose pseudoinverse of A, denoted by $A^\dagger$, is the unique matrix satisfying

$$\begin{aligned} AA^\dagger A &= A, \\ A^\dagger AA^\dagger &= A^\dagger, \\ (AA^\dagger)^\top &= AA^\dagger, \\ (A^\dagger A)^\top &= A^\dagger A. \end{aligned}$$

Assume that A has singular value decomposition $A = U\Sigma V^\top$. Let $\Sigma^\dagger$ be obtained by replacing each nonzero singular value σ of A by $1/\sigma$ and leaving zeros unchanged. Prove that

$$A^\dagger = V\Sigma^\dagger U^\top$$

satisfies the Moore–Penrose conditions.