39th Indian National Mathematical Olympiad – 2025

January 19, 2025

Time: 4.5 hours
Instructions:

- Calculators (in any form) and protractors are not allowed. Rulers and compasses are allowed.
- All questions carry equal marks. Maximum marks: 102.
- No marks will be awarded for stating an answer without justification.
- Answer to each question should start on a new page. Clearly indicate the question number.

Questions

1. Consider the sequence defined by $a_1 = 2$, $a_2 = 3$, and

$$a_{2k+2} = 2 + a_k + a_{k+1}, \quad a_{2k+1} = 2 + 2a_k$$

for all integers $k \geq 1$. Determine all positive integers n such that a_n is an integer.

2. Let $n \geq 2$ be a positive integer. The integers $1, 2, \ldots, n$ are written on a board. In a move, Alice can pick two integers written on the board $a \neq b$ such that $a+b$ is an even number, erase both a and b from the board, and write the number $\frac{a+b}{2}$ on the board instead. Find all n for which Alice can make a sequence of moves so that she ends up with only one number remaining on the board.
Note: When $n = 3$, Alice changes $(1, 2, 3)$ to (2) and cannot make any further moves.

3. Euclid has a tool called splitter which can only do the following two types of operations:

   (a) Given three non-collinear marked points X, Y, Z, it can draw the line which forms the interior angle bisector of $\angle XYZ$.

   (b) It can mark the intersection point of two previously drawn non-parallel lines.

   Suppose Euclid is only given three non-collinear marked points A, B, C in the plane. Prove that Euclid can use the splitter several times to draw the circle's centre passing through A, B, C.

4. Let $n \geq 3$ be a positive integer. Find the largest real number t_n as a function of n such that the inequality

$$\max(|a_1 + a_2|, |a_2 + a_3|, \ldots, |a_n - 1 + a_n|, |a_n + a_1|) \geq t_n \cdot \max(|a_1|, |a_2|, \ldots, |a_n|)$$

holds for all real numbers $a_1, a_2, \ldots, a_n$.

5. Greedy goblin Griphook has a regular 2000-gon, whose every vertex has a single coin. In a move, he chooses a vertex, removes one coin each from the two adjacent vertices, and adds one coin to the selected vertex, keeping the remaining coin for himself. He can only make such a move if both adjacent vertices have at least one coin. Griphook stops only when he cannot make any more moves. What is the **maximum** and **minimum** number of coins that he could have collected?

6. Let $b \geq 2$ be a positive integer. Anu has an infinite collection of notes with exactly $b - 1$ copies of a note worth $b^k - 1$ rupees, for every integer $k \geq 1$. A positive integer n is called payable if Anu can pay exactly $n^2 + 1$ rupees by using some collection of her notes. Prove that if there is a payable number, there are infinitely many payable numbers.