Trigonometry from the ashes of Calculus

Srikanth Pai,
Asst. Prof,
Madras School of Economics.

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Why write this article?

Our education system provides a multitude of choices for a young eager student. One of those choices, I have learnt recently, is called 'Applied Mathematics'. An inspection of the contents show much of geometry (trigonometry, coordinates, vectors, complex numbers) is removed but Calculus remains. I say that this is decision is very prudent. While students of applied mathematics lament that they have not learnt trigonometry in school, they do not know that Indian school trigonometry is mostly mindless algebra and it is very boring. However, the basic ideas surrounding its geometric meaning are very important for Fourier series and any student of science who wants to model periodic phenomena—especially macroeconomists passionate about understanding the business cycle.

I have written this note for my students who have taken applied mathematics. These students are extremely lucky, since learning trigonometric functions using differential equations is an absolute delight. When I was young, I was spoiled by the geometric route, and the differential equation route was confusing.

This note states and proves all the basic trigonometric relations starting from a single differential equation and the theorems of calculus. Secondly, it defines π from first principles using an integral and then works out the approximation 3.14. We also end up *proving* that this number π is the ratio of the circumference to the diameter of *any* circle in the world!

One wonders why I stop at 3.14—why not compute more digits? Well, it is futile to uncover more decimals since it will never end. Basically, π is irrational. Alas, proving its irrationality will take us far afield, but there is a gorgeous proof by Niven that you can look up.

I claim no novelty for this plan of attack. A whiff of these ideas is given in Simmon's text on differential equations and I suspect 18th century masters were well aware of this route [1]. However I have filled in the details and directed this movie. So sit tight and enjoy the ride.

1 The Fundamental Differential Equation

The Setup. Consider the second-order linear differential equation

$$y'' + y = 0. \tag{1}$$

[1] I say this because the same route with a slight turn carries us into the mysterious world of elliptic integrals, and then a little pondering leads to holomorphic differentials on complex algebraic curves and beyond.

This is one of the simplest nontrivial ODEs in mathematics. By the Picard existence-uniqueness theorem, given any initial conditions $y(0) = a$ and $y'(0) = b$, there exists a unique solution defined for all $x \in \mathbb{R}$.

From this single fact—and nothing else—we will construct all of trigonometry.

Definition 1.1. Let $S(x)$ denote the unique solution to the differential equation (1) satisfying the initial conditions

$$S(0) = 0, \quad S'(0) = 1. \tag{2}$$

We call S the *sine function.*

Definition 1.2. Let $C(x) := S'(x)$. We call C the *cosine function.*

Proposition 1.3. The function $C(x)$ satisfies the differential equation $y'' + y = 0$ with initial conditions

$$C(0) = 1, \quad C'(0) = 0. \tag{3}$$

Proof. Since $S'' + S = 0$, we have $C'(x) = S''(x) = -S(x)$. Differentiating again, $C''(x) = -S'(x) = -C(x)$, so $C'' + C = 0$.

By definition, $C(0) = S'(0) = 1$. And $C'(0) = -S(0) = 0$. □

Corollary 1.4. We have the fundamental derivative relations:

$$S'(x) = C(x), \quad C'(x) = -S(x). \tag{4}$$

Exercise 1.1. Verify that $S(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ and $C(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ satisfy the ODE and initial conditions. (Power series solution.)

Exercise 1.2. Show directly from the power series that $S'(x) = C(x)$ and $C'(x) = -S(x)$.

2 The Addition Formulas

The next goal is to establish the addition formulas, which express $S(x+u)$ and $C(x+u)$ in terms of $S(x), C(x), S(u), C(u)$. These are the *key* identities that govern all of trigonometry.

Theorem 2.1. For all $x, u \in \mathbb{R}$,

$$S(x+u) = S(x)C(u) + C(x)S(u). \tag{5}$$

Proof. Fix $u \in \mathbb{R}$ and define $y(x) := S(x+u)$. Since S solves $y'' + y = 0$, we have

$$y''(x) + y(x) = S''(x+u) + S(x+u) = 0.$$

At $x = 0$, the initial conditions are $y(0) = S(u)$ and $y'(0) = C(u)$.

Now consider the function

$$z(x) := S(x)C(u) + C(x)S(u).$$

We compute its derivatives:

$$\begin{aligned} z'(x) &= C(x)C(u) + (-S(x))S(u) = C(x)C(u) - S(x)S(u), \\ z''(x) &= -S(x)C(u) + (-C(x))S(u) = -[S(x)C(u) + C(x)S(u)] = -z(x). \end{aligned}$$

Thus $z'' + z = 0$, and at $x = 0$:

$$z(0) = S(0)C(u) + C(0)S(u) = 0 \cdot C(u) + 1 \cdot S(u) = S(u),$$

$$z'(0) = C(0)C(u) - S(0)S(u) = C(u) - 0 = C(u).$$

Both y and z satisfy the same ODE with the same initial conditions. By uniqueness, $y(x) = z(x)$ for all x, giving the desired result. □

Theorem 2.2. For all $x, u \in \mathbb{R}$,

$$C(x + u) = C(x)C(u) - S(x)S(u). \tag{6}$$

Proof. Differentiate equation (5) with respect to x:

$$\frac{d}{dx}S(x + u) = \frac{d}{dx}[S(x)C(u) + C(x)S(u)].$$

The left side is $C(x + u)$. The right side is

$$C(x)C(u) + (-S(x))S(u) = C(x)C(u) - S(x)S(u).$$

□

Corollary 2.3. For all $x \in \mathbb{R}$,

$$S(2x) = 2S(x)C(x), \tag{7}$$

$$C(2x) = C(x)^2 - S(x)^2. \tag{8}$$

Proof. Set $u = x$ in Theorems 2.1 and 2.2. □

Exercise 2.1. Prove the triple angle formulas: $S(3x) = 3S(x) - 4S(x)^3$ and $C(3x) = 4C(x)^3 - 3C(x)$.

Exercise 2.2. Using the addition formula, prove that $C(x - u) = C(x)C(u) + S(x)S(u)$.

Exercise 2.3. Prove the product-to-sum formulas: $S(x)C(u) = \frac{1}{2}[S(x + u) + S(x - u)]$.

3 The Pythagorean Identity

The most fundamental algebraic relation among S and C is the following identity, which will constrain these functions to the unit circle.

Theorem 3.1. For all $x \in \mathbb{R}$,

$$S(x)^2 + C(x)^2 = 1. \tag{9}$$

Proof. Rewrite the ODE $S''(x) + S(x) = 0$ as

$$\frac{dC}{dx}(x) = -S(x).$$

Multiply both sides by $C(x)$:

$$C(x)\frac{dC}{dx}(x) = -S(x)C(x).$$

Similarly, from $C'(x) = -S(x)$, multiply by $S(x)$:

$$S(x)\frac{dS}{dx}(x) = S(x)C(x).$$

Adding these two equations:

$$C(x)\frac{dC}{dx}(x) + S(x)\frac{dS}{dx}(x) = 0.$$

The left side is exactly $\frac{1}{2}\frac{d}{dx}[C(x)^2 + S(x)^2]$. Thus

$$\frac{d}{dx}[S(x)^2 + C(x)^2] = 0,$$

which means $S(x)^2 + C(x)^2$ is constant. At $x = 0$:

$$S(0)^2 + C(0)^2 = 0^2 + 1^2 = 1.$$

Therefore, $S(x)^2 + C(x)^2 = 1$ for all $x \in \mathbb{R}$. □

Corollary 3.2. For all $x \in \mathbb{R}$, $S(x) \in [-1, 1]$ and $C(x) \in [-1, 1]$.

Remark 3.3. Note the elegance of this argument: we never *assume* that S and C lie on the unit circle. Instead, the constraint $S^2 + C^2 = 1$ *emerges* purely from the dynamics of the ODE. This is a triumph of analytic thinking over geometric intuition.

Exercise 3.1. Prove that $[S(x)^2 + C(x)^2]' = 0$ directly from the formulas $S'(x) = C(x)$ and $C'(x) = -S(x)$.

Exercise 3.2. Show that $C(x)^2 - S(x)^2 = 2C(x)^2 - 1$, which relates the double angle formula to the Pythagorean identity.

4 Inverting the Integral and Defining π

From Theorem 3.1, we know $|S(x)| \leq 1$. Now we will invert the relationship between x and $S(x)$ to define the constant π analytically.

Definition 4.1. For $u \in (-1, 1)$, define

$$g(u) := \int_0^u \frac{dt}{\sqrt{1-t^2}}. \tag{10}$$

Lemma 4.2. For all $u \in (-1, 1)$, the integral $g(u) = \int_0^u \frac{dt}{\sqrt{1-t^2}}$ is well-defined. Moreover, g is continuously differentiable on $(-1, 1)$, strictly increasing, and $g'(u) = \frac{1}{\sqrt{1-u^2}} > 0$.

Proof. For any fixed $u \in (-1, 1)$, we have $u < 1$. On the interval $[0, u]$, the function $1 - t^2$ satisfies $1 - t^2 \geq 1 - u^2 > 0$. Therefore, the integrand $\frac{1}{\sqrt{1-t^2}}$ is continuous and bounded on $[0, u]$. By Riemann integration theory, the integral $g(u)$ exists. By the fundamental theorem of calculus, g is differentiable with

$$g'(u) = \frac{1}{\sqrt{1-u^2}}.$$

Since $1 - u^2 > 0$ for all $u \in (-1, 1)$, we have $g'(u) > 0$, making g strictly increasing on this interval. □

Lemma 4.3. The improper integral $\int_0^1 \frac{dt}{\sqrt{1-t^2}}$ converges.

Proof. For any $0 < \epsilon < 1$, the integral $\int_0^{1-\epsilon} \frac{dt}{\sqrt{1-t^2}}$ is a proper Riemann integral since the integrand is continuous and bounded on the closed interval $[0, 1-\epsilon]$.

For the tail integral, we use the limit comparison test. As $t \to 1^-$, we have

$$1 - t^2 = (1-t)(1+t) \to 0,$$

and the ratio

$$\frac{1/\sqrt{1-t^2}}{1/\sqrt{2(1-t)}} = \sqrt{\frac{2(1-t)}{(1-t)(1+t)}} = \sqrt{\frac{2}{1+t}} \to 1 \quad \text{as } t \to 1^-.$$

By the limit comparison test, $\int_{1-\epsilon}^1 \frac{dt}{\sqrt{1-t^2}}$ converges if and only if $\int_{1-\epsilon}^1 \frac{dt}{\sqrt{2(1-t)}}$ converges.

The latter integral is elementary:

$$\int_{1-\epsilon}^1 \frac{dt}{\sqrt{2(1-t)}} = \frac{1}{\sqrt{2}} \int_{1-\epsilon}^1 (1-t)^{-1/2}\, dt = \frac{1}{\sqrt{2}} \left[-2\sqrt{1-t}\right]_{1-\epsilon}^1 = \frac{2}{\sqrt{2}}\sqrt{\epsilon} < \infty.$$

Therefore, $\int_0^1 \frac{dt}{\sqrt{1-t^2}}$ converges. □

Definition 4.4. Define

$$\frac{\pi}{2} := \int_0^1 \frac{dt}{\sqrt{1-t^2}}. \tag{11}$$

By Lemma 4.3, this integral is well-defined and finite.

Definition 4.5. Define $\pi := 2 \cdot \frac{\pi}{2} = 2\int_0^1 \frac{dt}{\sqrt{1-t^2}}$.

Lemma 4.6. The function S is strictly increasing on the set $\{x \in \mathbb{R} : |S(x)| < 1\}$. Moreover, $S'(x) = \sqrt{1 - S(x)^2}$ for all such x.

Proof. From the Pythagorean identity (Theorem 3.1), $S(x)^2 + C(x)^2 = 1$. By Corollary 1.4, $S'(x) = C(x)$, so $S'(x)^2 = 1 - S(x)^2$.

For any x where $|S(x)| < 1$, we have $S'(x)^2 = 1 - S(x)^2 > 0$, so $S'(x) \neq 0$. Since $S'(0) = C(0) = 1 > 0$ and S is continuous, $S'(x) > 0$ for all x where $|S(x)| < 1$. Thus S is strictly increasing on the set $\{x : |S(x)| < 1\}$. □

Lemma 4.7. For all x such that $|S(x)| < 1$, we have $g(S(x)) = x$.

Proof. Differentiate the composition:

$$\frac{d}{dx} g(S(x)) = g'(S(x)) \cdot S'(x) = \frac{1}{\sqrt{1 - S(x)^2}} \cdot \sqrt{1 - S(x)^2} = 1.$$

Integrating, $g(S(x)) = x + C_0$. Since $g(S(0)) = g(0) = 0$, we have $C_0 = 0$. □

Lemma 4.8. The function S is odd: $S(-x) = -S(x)$ for all $x \in \mathbb{R}$.

Proof. Note that $S(-x)$ satisfies the ODE $y'' + y = 0$ with initial conditions $y(0) = S(0) = 0$ and $y'(0) = -S'(0) = -1$. By uniqueness of ODE solutions, $S(-x)$ equals the solution to this initial value problem. But $-S(x)$ also satisfies the same ODE and initial conditions. Therefore, $S(-x) = -S(x)$. □

Lemma 4.9. The function g is defined and continuous on the closed interval $[-1,1]$. It is strictly increasing on $(-1,1)$ since $g'(u) = \frac{1}{\sqrt{1-u^2}} > 0$ for all $u \in (-1,1)$. Moreover, g maps $[-1,1]$ bijectively onto $[-\pi/2, \pi/2]$.

Proof. For any $u \in (-1,1)$, the function g is defined and differentiable on that interval with $g'(u) > 0$, so g is strictly increasing. By continuity, g extends continuously to $[-1,1]$.

The endpoints are:

$$\begin{aligned} g(-1) &= \int_0^{-1} \frac{dt}{\sqrt{1-t^2}} = -\int_0^1 \frac{dt}{\sqrt{1-t^2}} = -\frac{\pi}{2}, \\ g(1) &= \int_0^1 \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{2}. \end{aligned}$$

Since g is continuous and strictly increasing on $[-1,1]$ with $g(-1) = -\pi/2$ and $g(1) = \pi/2$, by the intermediate value theorem, g maps $[-1,1]$ bijectively onto $[-\pi/2, \pi/2]$. $\square$

Lemma 4.10. We have $S(\pi/2) = 1$ and $S(-\pi/2) = -1$.

Proof. By Lemma 4.7, $g(S(x)) = x$ for all x where $|S(x)| < 1$.

Applying this identity at $x = \pi/2$:

$$g(S(\pi/2)) = \pi/2.$$

But by Lemma 4.9, $g(1) = \pi/2$. Since g is one-to-one on $[-1,1]$ (being strictly increasing), we have $S(\pi/2) = 1$.

By Lemma 4.8, $S(-\pi/2) = -S(\pi/2) = -1$. $\square$

Theorem 4.11. We have

$$S\left(\frac{\pi}{2}\right) = 1, \tag{12}$$

$$S\left(-\frac{\pi}{2}\right) = -1, \tag{13}$$

$$C\left(\frac{\pi}{2}\right) = 0. \tag{14}$$

Proof. By Lemma 4.10, $S(\frac{\pi}{2}) = 1$. From the Pythagorean identity, $C(\frac{\pi}{2})^2 = 1 - S(\frac{\pi}{2})^2 = 0$, so $C(\frac{\pi}{2}) = 0$.

For the third claim, observe that both S and g satisfy their defining properties with respect to the substitution $x \mapsto -x$. By the uniqueness of the ODE solution, $S(-x)$ solves the same ODE with initial conditions $S(-0) = 0$ and $S'(-0) = -C(0) = -1$. Thus $S(-x) = -S(x)$. Therefore, $S(-\frac{\pi}{2}) = -S(\frac{\pi}{2}) = -1$. $\square$

Theorem 4.12. The functions S and g are inverses on their respective domains:

$$g(S(x)) = x \quad \text{for all } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \tag{15}$$

$$S(g(u)) = u \quad \text{for all } u \in (-1,1). \tag{16}$$

Consequently, $S : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$ is a bijection, and both functions are strictly monotone.

Proof. Lemma 4.7 established $g(S(x)) = x$ for all x where $|S(x)| < 1$. By Theorem 4.11, S achieves the boundary values $S(\pm\frac{\pi}{2}) = \pm 1$. Therefore, Lemma 4.7 extends to the closed interval:

$$g(S(x)) = x \quad \text{for all } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Since g is strictly increasing on $(-1, 1)$ (Lemma 4.2), and the above relationship holds, the inverse relationship $S(g(u)) = u$ follows immediately for all $u \in (-1, 1)$ in the range of S on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The range of S on this interval is exactly $[-1, 1]$ (by continuity and the boundary values).

Thus S is the inverse of g on the proper domains, establishing the bijection $S : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$. □

Corollary 4.13. We have

$$S(\pi) = 0, \quad C(\pi) = -1, \tag{17}$$
$$S(2\pi) = 0, \quad C(2\pi) = 1. \tag{18}$$

Proof. Use the addition formulas. For example,

$$C(\pi) = C(\pi/2 + \pi/2) = C(\pi/2)C(\pi/2) - S(\pi/2)S(\pi/2) = 0 - 1 = -1.$$

And $S(\pi) = S(\pi/2 + \pi/2) = S(\pi/2)C(\pi/2) + C(\pi/2)S(\pi/2) = 0$. □

Exercise 4.1. Explain why the integral $\int_0^1 \frac{dt}{\sqrt{1-t^2}}$ converges, despite the singularity at $t = 1$. (Hint: near $t = 1$, the integrand behaves like $\frac{1}{\sqrt{2(1-t)}}$.)

Exercise 4.2. Show that S and C are odd and even, respectively. That is, $S(-x) = -S(x)$ and $C(-x) = C(x)$.

Exercise 4.3. Compute $S(3\pi/2)$ and $C(3\pi/2)$ using addition formulas.

5 Bounds on π via Integral Inequalities

The elegant Dalzell integral (1944) provides a rigorous and computable bound on π.

Lemma 5.1. We have

$$\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}. \tag{19}$$

Proof. Let $x = \frac{S(u)}{C(u)}$ where $u \in [0, \pi/4]$. Then:

$$\frac{dx}{du} = \frac{S'(u)C(u) - S(u)C'(u)}{C(u)^2} = \frac{C(u) \cdot C(u) - S(u) \cdot (-S(u))}{C(u)^2} = \frac{C(u)^2 + S(u)^2}{C(u)^2} = \frac{1}{C(u)^2}.$$

Also:

$$1 + x^2 = 1 + \frac{S(u)^2}{C(u)^2} = \frac{C(u)^2 + S(u)^2}{C(u)^2} = \frac{1}{C(u)^2}.$$

When $u = 0$: $x = 0$. When $u = \pi/4$: $S(\pi/4) = C(\pi/4)$ (by the Pythagorean identity and symmetry), so $x = 1$.

Substituting:

$$\int_0^1 \frac{dx}{1+x^2} = \int_0^{\pi/4} \frac{1}{1/C(u)^2} \cdot \frac{1}{C(u)^2}\, du = \int_0^{\pi/4} 1\, du = \frac{\pi}{4}.$$

□

Lemma 5.2. We have

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\, dx = \frac{22}{7} - \pi. \tag{20}$$

Proof. Expand the numerator:

$$x^4(1-x)^4 = x^4 - 4x^5 + 6x^6 - 4x^7 + x^8.$$

Divide by $1+x^2$ using polynomial long division:

$$\frac{x^4 - 4x^5 + 6x^6 - 4x^7 + x^8}{1+x^2} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}.$$

Integrate both sides from 0 to 1:

$$\begin{aligned}
\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\, dx &= \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}\right) dx \\
&= \left[\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4x^3}{3} + 4x\right]_0^1 - 4\int_0^1 \frac{dx}{1+x^2} \\
&= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - 4 \cdot \frac{\pi}{4} \quad \text{(by Lemma 5.1)} \\
&= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - \pi.
\end{aligned}$$

Computing the rational part:

$$\frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 = \frac{1}{7} + 5 - 2 = \frac{22}{7}.$$

Therefore, $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\, dx = \frac{22}{7} - \pi$. □

Proposition 5.3. The integral $I := \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\, dx$ satisfies

$$\frac{1}{1260} < I < \frac{1}{630}. \tag{21}$$

Proof. The integrand is continuous and nonnegative on $[0,1]$. For a lower bound, we replace the denominator $1+x^2$ with the larger value 2 (achieved at $x=1$):

$$I > \int_0^1 \frac{x^4(1-x)^4}{2}\, dx = \frac{1}{2}\int_0^1 x^4(1-x)^4\, dx.$$

The integral $\int_0^1 x^4(1-x)^4\, dx$ can be computed using the beta function. With exponents $a=5, b=5$:

$$\int_0^1 x^4(1-x)^4\, dx = \frac{\Gamma(5)\Gamma(5)}{\Gamma(10)} = \frac{4! \cdot 4!}{9!} = \frac{576}{362880} = \frac{1}{630}.$$

Thus $I > \frac{1}{1260}$.

For an upper bound, we replace the denominator with the smaller value 1 (achieved at $x = 0$):

$$I < \int_0^1 x^4(1-x)^4\,dx = \frac{1}{630}.$$

□

Corollary 5.4. From Lemma 5.2 and Proposition 5.3,

$$\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}. \tag{22}$$

In decimal form,

$$3.1412698 < \pi < 3.1420635. \tag{23}$$

Therefore, to two decimal places, $\pi \approx 3.14$.

Exercise 5.1. Verify the bounds in Corollary 5.4 numerically using the known value $\pi \approx 3.14159265$.

Exercise 5.2. Show that $S(\pi/4) = C(\pi/4)$ using the Pythagorean identity and the addition formula for $S(2u)$.

6 Periodicity and Angle Addition Identities

We now establish the periodicity of S and C, which shows that these functions repeat with period 2π.

Theorem 6.1. For all $x \in \mathbb{R}$,

$$S(x + 2\pi) = S(x), \quad C(x + 2\pi) = C(x). \tag{24}$$

That is, both S and C have period 2π.

Proof. We use the addition formulas repeatedly. From Corollary 4.13, we know $S(2\pi) = 0$ and $C(2\pi) = 1$. Thus,

$$S(x + 2\pi) = S(x)C(2\pi) + C(x)S(2\pi) = S(x) \cdot 1 + C(x) \cdot 0 = S(x),$$
$$C(x + 2\pi) = C(x)C(2\pi) - S(x)S(2\pi) = C(x) \cdot 1 - S(x) \cdot 0 = C(x).$$

□

Corollary 6.2. For all $x \in \mathbb{R}$,

$$S(\pi - x) = S(x), \quad C(\pi - x) = -C(x). \tag{25}$$

Proof. Using the addition formula:

$$S(\pi - x) = S(\pi)C(x) - C(\pi)S(x) = 0 \cdot C(x) - (-1) \cdot S(x) = S(x),$$
$$C(\pi - x) = C(\pi)C(x) + S(\pi)S(x) = (-1) \cdot C(x) + 0 \cdot S(x) = -C(x).$$

□

Corollary 6.3. For all $x \in \mathbb{R}$,

$$S(\pi/2 - x) = C(x), \quad C(\pi/2 - x) = S(x). \tag{26}$$

Proof. Using the addition formula:

$$S(\pi/2 - x) = S(\pi/2)C(x) - C(\pi/2)S(x) = 1 \cdot C(x) - 0 \cdot S(x) = C(x),$$
$$C(\pi/2 - x) = C(\pi/2)C(x) + S(\pi/2)S(x) = 0 \cdot C(x) + 1 \cdot S(x) = S(x).$$

□

Exercise 6.1. Prove the half-angle formulas: $S(x/2) = \sqrt{\frac{1-C(x)}{2}}$ (for appropriate branch).

Exercise 6.2. Prove $S(x + \pi) = -S(x)$ and $C(x + \pi) = -C(x)$ (period π for signed periodicity).

Exercise 6.3. Show that $S(\pi/4) = C(\pi/4) = 1/\sqrt{2}$ using the double-angle formula.

7 The Unit Circle Parametrization

We now arrive at the geometric interpretation of S and C: they parametrize the unit circle in $\mathbb{R}^2$.

Theorem 7.1. The map $\Phi : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\Phi(x) := (C(x), S(x)) \tag{27}$$

is a continuous bijection from $[0, 2\pi)$ onto the unit circle $\{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = 1\}$. Moreover, Φ is periodic with period 2π.

Proof. By the Pythagorean identity (Theorem 3.1), every point $(C(x), S(x))$ lies on the unit circle.

For injectivity on $[0, 2\pi)$: Since S is strictly increasing on $[0, \pi/2]$, C is strictly decreasing on $[0, \pi]$ (as $C' = -S < 0$), and we can establish injectivity by tracking the quadrant behavior on each interval $[k\pi/2, (k + 1)\pi/2]$.

For surjectivity: any point (a, b) on the unit circle has $a^2+b^2 = 1$, so $|b| \leq 1$. By the intermediate value theorem and monotonicity of S on $[-\pi/2, \pi/2]$, there exists a unique $x_0 \in [-\pi/2, \pi/2]$ with $S(x_0) = b$. By the addition formulas and periodicity, we can find $x \in [0, 2\pi)$ with both $S(x) = b$ and $C(x) = a$. Thus every point on the unit circle is in the image.

Periodicity follows from Theorem 6.1. □

Remark 7.2. The parameter x in the map $\Phi(x) = (C(x), S(x))$ has a physical meaning: it is the *arc length* along the unit circle measured counterclockwise from the point $(1, 0)$.

Exercise 7.1. For which values of $x \in [0, 2\pi)$ is $C(x) = 0$? Is $S(x) = 0$?

Exercise 7.2. Show that Φ restricted to $[0, \pi/2]$ is a bijection onto the first quadrant arc of the unit circle.

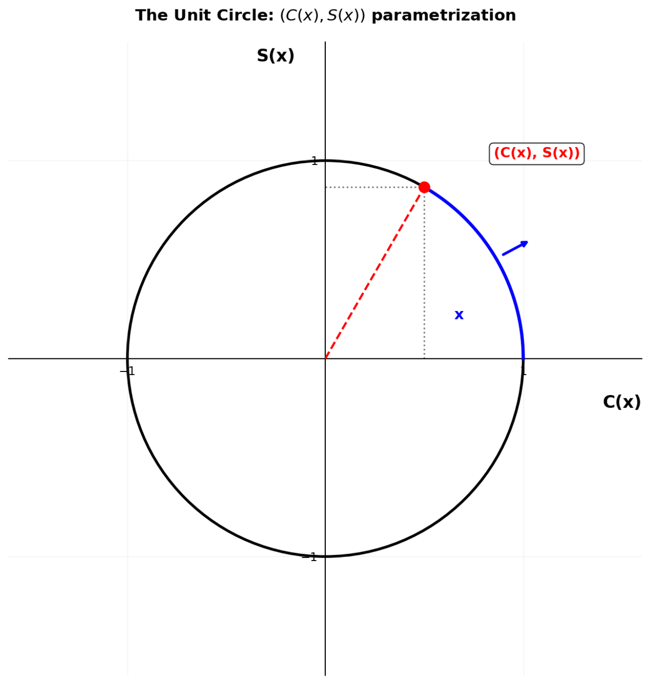

Figure 1: The unit circle parametrized by $(C(x), S(x))$ for $x \in [0, 2\pi)$.

8 Arc Length and the Circumference Formula

Finally, we derive the classical formula for the circumference of a circle using only calculus and our analytic definitions of S and C.

We have shown that a point on the unit circle can be represented as $(C(x), S(x))$, then the natural question is what is 'x' from the point of view of the circle. We claim that it is the arc length of the arc starting from $(0, 1)$ to the given point moving anti-clockwise.

The main tool used is that the velocity vector is rate of change of displacement and the magnitude of velocity vector is speed. Further speed into time is distance travelled.

Proposition 8.1. The curve $(C(t), S(t))$ for $t \in [a, b]$ has velocity vector

$$\frac{d}{dt}(C(t), S(t)) = (-S(t), C(t)). \tag{28}$$

Proof. By Corollary 1.4, $\frac{d}{dt}C(t) = C'(t) = -S(t)$ and $\frac{d}{dt}S(t) = S'(t) = C(t)$. □

Corollary 8.2. The speed (magnitude of the velocity vector) is

$$\left\| \frac{d}{dt}(C(t), S(t)) \right\| = \sqrt{S(t)^2 + C(t)^2} = \sqrt{1} = 1. \tag{29}$$

Proof. By the Pythagorean identity, $S(t)^2 + C(t)^2 = 1$. □

Theorem 8.3. The arc length of the unit circle (i.e., the circumference of a circle of radius 1) is 2π.

Proof. The unit circle is parametrized by $(C(t), S(t))$ for $t \in [0, 2\pi)$. The arc length formula gives

$$L = \int_0^{2\pi} \left\| \frac{d}{dt}(C(t), S(t)) \right\| dt = \int_0^{2\pi} 1\, dt = 2\pi.$$

$\square$

Theorem 8.4. For a circle of radius $r > 0$ centered at the origin, the circumference is

$$\text{Circumference} = 2\pi r. \tag{30}$$

Proof. A circle of radius r can be parametrized as $(rC(t), rS(t))$ for $t \in [0, 2\pi)$. The velocity vector is

$$\frac{d}{dt}(rC(t), rS(t)) = r(-S(t), C(t)).$$

The speed is

$$\|r(-S(t), C(t))\| = r\sqrt{S(t)^2 + C(t)^2} = r \cdot 1 = r.$$

Therefore, the arc length is

$$L = \int_0^{2\pi} r\, dt = r \cdot 2\pi = 2\pi r.$$

$\square$

Remark 8.5. Notice that this derivation of the circumference formula is *entirely analytic*. We made no appeal to geometric properties, similar triangles, or any classical geometric construction. The formula emerged purely from:

1. The ODE $y'' + y = 0$,
2. Calculus (differentiation and integration),
3. The Pythagorean identity $S^2 + C^2 = 1$,
4. The arc length formula from multivariable calculus.

The geometry is a *consequence*, not a premise.

Exercise 8.1. Compute the arc length of the curve $(C(t), S(t))$ for $t \in [0, \pi/2]$. Verify that it equals $\pi/2$.

Exercise 8.2. Show that a circle of radius r with arc length from $t = t_1$ to $t = t_2$ has arc length $r(t_2 - t_1)$.

Exercise 8.3. If the circumference of a circle is $2\pi r$, express the area of the circle as an integral. (Hint: $A = \int_0^r 2\pi\rho\, d\rho$ where ρ is the radial distance.)

Exercise 8.4. Consider the spiral $(t \cdot C(t), t \cdot S(t))$ for $t \in [0, 2\pi]$. Compute its arc length.

Conclusion

We have reconstructed trigonometry entirely from the differential equation $y'' + y = 0$. Starting with nothing but calculus and the existence-uniqueness theorem for ODEs, we derived:

1. The definitions and basic properties of $S(x)$ and $C(x)$,
2. The addition formulas,
3. The Pythagorean identity,
4. The constant π and its numerical value,
5. Periodicity and angle addition formulas,
6. The parametrization of the unit circle,
7. The circumference formula for any circle.

All of this emerges from a single, elegant differential equation. This is the power of analytic thinking: geometry, far from being a foundation, becomes a natural consequence of algebraic relationships and calculus.

A Summary of Key Formulas

Identity	**Formula**
Differential equation	$S'' + S = 0, \quad C'' + C = 0$
Derivatives	$S'(x) = C(x), \quad C'(x) = -S(x)$
Addition formulas	$S(x+u) = S(x)C(u) + C(x)S(u)$ $C(x+u) = C(x)C(u) - S(x)S(u)$
Pythagorean	$S(x)^2 + C(x)^2 = 1$
Double angle	$S(2x) = 2S(x)C(x)$ $C(2x) = C(x)^2 - S(x)^2$
Special values	$S(\pi/2) = 1, \quad C(\pi/2) = 0$ $S(\pi) = 0, \quad C(\pi) = -1$
π definition	$\pi = 2\int_0^1 \frac{dt}{\sqrt{1-t^2}}$
Periodicity	$S(x+2\pi) = S(x), \quad C(x+2\pi) = C(x)$
Supplementary	$S(\pi - x) = S(x), \quad C(\pi - x) = -C(x)$
Complementary	$S(\pi/2 - x) = C(x), \quad C(\pi/2 - x) = S(x)$
Circumference	Circumference of radius $r = 2\pi r$