A Median Voter Theorem with Voter Abstention

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Abstract. We study a two-party spatial competition model that extends the Hotelling–Downs framework by allowing voter abstention. Voters share a common voting window size and participate only if at least one party lies within that window. Parties choose policy positions to maximize their vote-share margin.
We show that if a pure-strategy Nash equilibrium exists, both parties must choose the same policy position that generalizes the median voter principle. In the classical model, the median splits the total electorate into two equal halves; here, the equilibrium position must split the participating voters within a fixed voting window into two equal parts. We then give necessary and sufficient conditions on the voter distribution for the existence of pure-strategy Nash equilibria. We identify broad classes of voter distributions under which pure-strategy equilibria exist.

Keywords: Spatial competition · Abstention · Median voter theorem · Nash equilibrium

1 Introduction

In the Hotelling model of spatial competition [8], refined for elections by Downs [4], voters have preferences over a one-dimensional policy space and always vote for the closer candidate. Candidates choose policy positions to maximize the chance of winning. With full turnout, maximizing absolute vote share and maximizing the vote share margin are equivalent objectives. Under these assumptions, any deviation from the median voter's position loses more voters on one side than it gains on the other. As a result, in a two-party election both candidates locate at the median of the voter distribution.

A central assumption behind the median positioning result is full voter turnout, which is often unrealistic. In real elections, participation is not unconditional. Voting involves informational, administrative, and physical costs, which can reduce participation [1, 11]. In addition, voters may abstain when no party lies sufficiently close to their ideal point [12].

Several recent papers model incomplete turnout by assuming that voters participate only if at least one candidate lies within a limited attraction interval.

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Following Feldman et al.[6] and Shen and Wang[10], voters abstain when no party is sufficiently close. In these models, voters who participate choose randomly among parties within their attraction interval. Feldman et al.[6] study participation-maximization and related objectives, typically under a uniform voter distribution, while Shen and Wang[10] extend the analysis to general distributions and show that pure equilibria need not always exist under winner-take-all objectives. Related work allows attraction intervals to be stochastic [3] or imposes unit-demand constraints [7], while retaining absolute vote-share objectives. In contrast, we use the attraction interval only to determine participation: voters who participate vote for the nearest party within range. Since we assume a continuous voter distribution, ties occur on a set of measure zero and do not affect payoffs.

Empirical and theoretical work in political science and economics shows that the size of a victory margin influences post-election outcomes and strategic behavior: candidates and parties interpret larger margins as a stronger mandate, which can affect policy choices, bargaining power, and political legitimacy, and models where "mandate" influences incentives have been studied in the literature [2, 5]. Thus, we are interested in vote-share margin maximization.

Our model, presented in Section 2, considers voters who vote the nearest party in a window and abstain otherwise. The parties maximise the vote share margin. While pure equilibria are not guaranteed for all distributions under this objective, we generalise the median voting result.

Our first result shows that if a pure-strategy Nash equilibrium exists, then both candidates must locate at a conditional median of the voter distribution (Theorem 1), extending the classical median voter principle to a limited attraction interval setting. We show that conditional medians always exist for continuous voter distributions.

Existence of a conditional median, however, is not sufficient for equilibrium. Whether a candidate can profitably deviate depends on how voter mass is distributed within and across attraction windows. We formalize this requirement through a notion of window dominance, which captures both global and local incentives to deviate. Our main characterisation result shows that a symmetric pure-strategy Nash equilibrium exists if and only if a conditional median is window-dominant (Theorem 2). This yields transparent sufficient conditions for equilibrium existence in a broad class of voter distributions.

Section 2 presents the model. Section 3 derives the conditional median condition. Section 4 studies window dominance for standard classes of voter distributions. Section 5 characterizes symmetric pure-strategy Nash equilibria. Section 6 concludes. All the proofs are in the appendix.

2 Model Framework and Preliminaries

We consider a continuum of voters with ideological positions $t \in [0, 1]$, distributed according to a continuous cumulative distribution function F. Each voter has a limited voting radius $c > 0$ that presents a limited attraction window for a party. More precisely: A voter at t votes only if at least one party lies in $[t-c, t+c]$; otherwise the voter abstains. Among admissible parties, the voter votes for the closest one. If both parties are equidistant, the voter randomizes uniformly.

Two parties, A and B, simultaneously choose ideological positions $a, b \in [0,1]$. Let T be a random variable with a *continuous* cumulative distribution function F, representing the ideological position of a randomly sampled voter. Since ties occur on a set of measure zero, the tie-breaking rule does not affect payoffs. For any strategy profile (a, b), the expected fraction of the vote share gained by A is defined as

$$V_A(a,b) := \Pr\big(|T-a| \leq \min\{|T-b|, c\}\big),$$

with $V_B(a,b)$ defined symmetrically.

Parties maximize the vote share margin. Payoffs are given by

$$P_A(a,b) = V_A(a,b) - V_B(a,b), \qquad P_B(a,b) = V_B(a,b) - V_A(a,b).$$

The game is symmetric and zero-sum.

A game $G(F,c)$ is fully specified by the voter distribution F and the attraction radius c, which are common knowledge. A Nash equilibrium is a strategy profile (a^*, b^*) such that no party can profitably deviate unilaterally.

Definition 1. *A strategy profile (a^*, b^*) is a* ***Pure Strategy Nash equilibrium*** *(PSNE) if*

$$P_A(a^*,b^*) \geq P_A(a,b^*) \quad \forall a \in [0,1], \qquad P_B(a^*,b^*) \geq P_B(a^*,b) \quad \forall b \in [0,1].$$

Further, if $a^* = b^*$, we say that the PSNE is symmetric.

The following fact is well known in the literature on symmetric zero-sum games and follows from [9].

Proposition 1. *Consider a two-player symmetric zero-sum game.*

1. *If (x,y) is a pure-strategy Nash equilibrium, then (x,x) and (y,y) are also pure-strategy Nash equilibria.*
2. *Conversely, if (x,x) and (y,y) are pure-strategy Nash equilibria, then (x,y) and (y,x) are also pure-strategy Nash equilibria.*

In particular, the set of pure-strategy Nash equilibria is rectangular: it is the Cartesian product of the sets of pure equilibrium strategies of the two players.

This proposition allows us to restrict attention to symmetric PSNEs. To analyze such equilibria, we study the mass of voters captured within attraction windows.

Definition 2. *Let $I \subset [0,1]$ be an interval of length $|I| > 0$. The* window mass *of I is*

$$w(I) := F(\sup I) - F(\inf I).$$

The window mass density *of* I *is*

$$\rho(I) := \frac{w(I)}{|I|}.$$

Fix $c > 0$. *The* c-window mass *at* x *is defined as*

$$w(x, c) := w([x - c, x + c]).$$

3 Conditional Median Voter Theorem

We characterize pure strategy Nash equilibria of $G(F, c)$. Since the game is symmetric and zero-sum, any Nash equilibrium yields zero payoff to both players.

To refine the necessary conditions for an equilibrium, we must consider how a candidate's position is constrained by the specific boundaries of their attraction.

Definition 3 (Conditional Median). *Let* F *be a cumulative distribution function on* $[0, 1]$ *and fix* $c > 0$. *A point* $a \in [0, 1]$ *is a* c-conditional median *of* F *if:*

$$F(a + c) - F(a) = F(a) - F(a - c) \tag{1}$$

If a real number is a c-conditional median for some positive real c, it is called a conditional median of the distribution F.

Let $CM(F, c)$ *and* $CM(F)$ *denote the set of c-conditional medians and conditional medians of the distribution* F *respectively.*

We quickly explain why the definition is natural.

Proposition 2 (Conditional median interpretation). *Let* T *be a random variable with cumulative distribution function* F. *Fix* $c > 0$ *and* $a \in [0, 1]$, *and define the conditional distribution*

$$F_a(x) = \Pr(T \le x \mid T \in [a - c, a + c]), \qquad x \in [a - c, a + c].$$

Then a *satisfies*

$$F(a + c) - F(a) = F(a) - F(a - c)$$

if and only if a *is a median of the conditional distribution* F_a.

Figure 1 illustrates the conditional median graphically: the point a divides the voter mass within the attraction window $[a - c, a + c]$ into two equal parts.

Theorem 1. *Let* F *be a continuous cumulative distribution function on* $[0, 1]$. *If* (a^*, b^*) *is a pure strategy Nash Equilibrium of the game* $G(F, c)$, *then* a^* *and* b^* *must be conditional medians of distribution* F.

Introducing voter abstention replaces the classical median voting principle with a conditional median criterion: parties must locate at positions that balance voter mass within their attraction windows. For sufficiently large values of c, this condition coincides with the median of the full distribution.

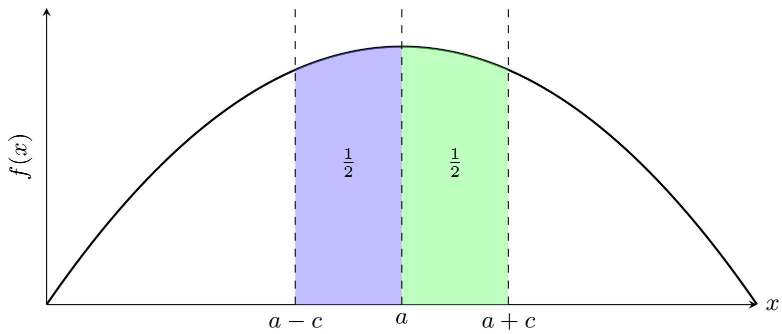

Fig. 1. The conditional median a splits the voter mass within the window $[a-c, a+c]$ into two equal parts.

Corollary 1 (Reduction to the median voter).
Let F be a continuous cumulative distribution function on $[0,1]$ with median m, i.e. $F(m) = \frac{1}{2}$. For $c \geq \max\{m,\, 1-m\}$, the point m satisfies the conditional median equation.

$$F(a+c) - F(a) = F(a) - F(a-c)$$

Next we address the question of whether conditional medians exist for every distribution. The following lemma establishes that the conditional median equation always admits a solution.

Proposition 3. *Let F be a continuous cumulative density function. Then there exists a c-conditional median for F, i.e. $CM(F, c) \neq \phi$*

4 Window dominance of a distribution

Now we isolate a property of a distribution that helps us characterise symmetric pure equilibria in Section 5.

Definition 4 (Window Dominance). *Fix $c > 0$ and let F be a distribution on $[0,1]$. Let $a \in [0,1]$ and define $I^* = [a-c, a+c]$.*

1. *(**Global c-dominance at** a) We say that F is* globally c-dominant at *a if for every interval $J \subset [0,1]$ of length $2c$ with $J \cap I^* = \varnothing$,*

$$\rho(I^*) \geq \rho(J).$$

2. *(**Local c-dominance at** a) We say that F is* locally c-dominant at *a if for every $\delta \in (0, 2c]$,*

$$\rho([a+c, a+c+\delta]) \;\leq\; \rho([a, a+\tfrac{\delta}{2}]),$$

and

$$\rho([a-c-\delta, a-c]) \;\leq\; \rho([a-\tfrac{\delta}{2}, a]).$$

*We say that F is c-*window dominant at *a if it is both globally and locally c-dominant at a.*

We now identify two broad classes of functions that are window dominant.

Proposition 4. *Let f be a continuous density on $[0,1]$ that is symmetric and unimodal with mode $a \in (0,1)$. Fix $c > 0$ such that $[a-c, a+c] \subset [0,1]$. Then F is c-window dominant at a.*

Proposition 5. *Fix $c > 0$ and let f be a continuous density on $[0,1]$.*

1. *If f is non-decreasing, then F is c-window dominant at a point $a \in [1-c, 1]$.*
2. *If f is non-increasing, then F is c-window dominant at a point $a \in [0, c]$.*

5 Characterisation of pure symmetric equilibria

In this section we state the characterisation theorem: window dominance at the conditional median is the condition that characterises symmetric pure-strategy equilibria.

Theorem 2 (Characterisation of PSNE). *Fix $c > 0$ and a continuous distribution F. A symmetric pure-strategy Nash equilibrium exists for $G(F, c)$ at (a, a) if and only if a is a conditional median which is window-dominant.*

We quickly explain the intuition behind the proof. When both players locate at the same conditional median, each player's vote share is obtained from exactly one side of that point. Consider a unilateral deviation by one player. If the deviation places the new attraction window disjoint from the original one, the payoff comparison reduces to a comparison of window masses; this corresponds to global window dominance.

If instead the attraction windows overlap, the situation is as illustrated in Figure 2. The deviating player gains a voter mass G at the boundary of the attraction window, but loses a voter mass L near the conditional median. Because voters near the median are split at the indifference point, the loss L enters the vote-share margin twice, while the gain G enters only once. Thus a deviation is unprofitable if and only if $G \leq 2L$, which is exactly the local window dominance condition at the conditional median.

Even for unimodal voter distributions, the symmetric middle position need not be stable under bounded tolerance. While high voter density near the center attracts both candidates, locating too close forces them to split this mass. A small unilateral deviation can allow one candidate to capture most of the central voters while losing only a small amount of peripheral support. This creates a local incentive to deviate that can rule out pure strategy Nash equilibria.

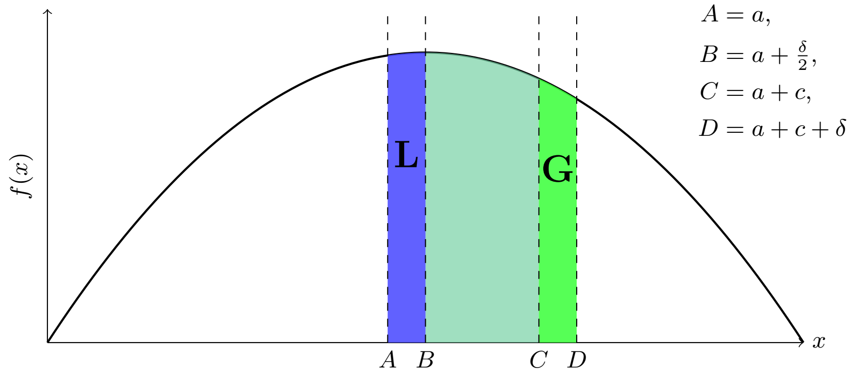

Fig. 2. Local deviation at the conditional median a. The blue window represents the original attraction interval $[a, a + c]$, while the green window represents the shifted interval $[a + \frac{\delta}{2}, a + \delta + c]$. Their overlap is shaded naturally. Mass **L** lost near the center is counted twice in the vote-share margin, whereas mass **G** gained at the boundary is counted once.

6 Conclusion

We study two-party spatial competition when voters participate only if at least one party lies within a fixed distance of their ideal point, and parties maximize the vote-share margin. This setting modifies the classical median voter result. If a pure-strategy Nash equilibrium exists, both parties must locate at a conditional median of the voter distribution. Such points always exist for continuous distributions, but equilibrium existence is no longer guaranteed for general voter distributions. This contrasts with the Hotelling–Downs model, where existence and uniqueness are automatic.

We show that equilibrium existence depends on how voter mass is distributed within attraction windows. Even at a conditional median, local deviations may be profitable when voter density is uneven across the window. We formalize this mechanism through window dominance and use it to characterize symmetric pure-strategy equilibria.

Several questions remain open. First, the structure of the set of conditional medians as the attraction interval length varies is not yet well understood. Second, while symmetric unimodality and monotonicity provide sufficient conditions for window dominance, it remains an open question how far these conditions can be relaxed. Identifying alternative shape restrictions on the voter distribution that ensure window dominance is a natural direction for future work. Finally, the model raises empirical questions about candidate positioning when voter participation depends on ideological distance. Exploring these implications would require combining the theory with independent evidence on voter abstention and tolerance, which we leave for future work.

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Appendix

A Proof of Proposition 2

Proof. Fix $c > 0$ and $a \in [0, 1]$. Let

$$F_a(x) = \Pr(T \le x \mid T \in [a-c, a+c]), \qquad x \in [a-c, a+c].$$

For any $x \in [a-c, a+c]$, by the definition of conditional probability,

$$\begin{aligned} F_a(x) &= \frac{\Pr(T \le x,\ T \in [a-c, a+c])}{\Pr(T \in [a-c, a+c])} \\ &= \frac{F(x) - F(a-c)}{F(a+c) - F(a-c)}. \end{aligned}$$

The point a is a median of the conditional distribution F_a if and only if

$$F_a(a) = \tfrac{1}{2}.$$

Substituting the expression above yields

$$\frac{F(a) - F(a-c)}{F(a+c) - F(a-c)} = \tfrac{1}{2},$$

which is equivalent to

$$F(a) - F(a-c) = \tfrac{1}{2}\big(F(a+c) - F(a-c)\big).$$

Rearranging gives

$$F(a+c) - F(a) = F(a) - F(a-c).$$

Thus a satisfies the conditional median equation if and only if a is a median of the conditional distribution F_a.

B Proof of Theorem 1

Let (a^*, b^*) be a pure strategy Nash equilibrium of $G(F, c)$. Since the game is symmetric and zero-sum, equilibrium payoffs satisfy

$$P_A(a^*, b^*) = P_B(a^*, b^*) = 0.$$

Suppose, toward a contradiction, that a^* is not a conditional median. Without loss of generality, assume

$$F(a^* + c) - F(a^*) \ > \ F(a^*) - F(a^* - c). \tag{2}$$

Consider a deviation by player B to

$$b = a^* + \varepsilon,$$

where $\varepsilon > 0$ is sufficiently small. The indifference point between a^* and b is

$$m(\varepsilon) = \frac{a^* + b}{2} = a^* + \frac{\varepsilon}{2}.$$

For ε small enough, the attraction windows overlap, and voters in

$$[m(\varepsilon),\, a^* + \varepsilon + c]$$

vote for player B, while voters in

$$[a^* - c,\, m(\varepsilon)]$$

vote for player A.

Therefore the vote shares are

$$\begin{aligned} V_B(a^*, b) &= F(a^* + \varepsilon + c) - F\big(a^* + \tfrac{\varepsilon}{2}\big)\,, \\ V_A(a^*, b) &= F\big(a^* + \tfrac{\varepsilon}{2}\big) - F(a^* - c). \end{aligned}$$

The payoff difference for player B is thus

$$\begin{aligned} \varDelta V(\varepsilon) &= V_B(a^*, b) - V_A(a^*, b) \\ &= \big[F(a^* + \varepsilon + c) - F(a^* + \varepsilon/2)\big] - \big[F(a^* + \varepsilon/2) - F(a^* - c)\big]. \end{aligned} \tag{3}$$

By continuity of F,

$$\lim_{\varepsilon \to 0^+} \varDelta V(\varepsilon) = \big[F(a^* + c) - F(a^*)\big] - \big[F(a^*) - F(a^* - c)\big].$$

By assumption (2), this limit is strictly positive. Hence there exists $\delta > 0$ such that

$$\varDelta V(\varepsilon) > 0 \quad \text{for all } \varepsilon \in (0, \delta).$$

Thus player B has a strictly profitable unilateral deviation from (a^*, b^*), contradicting the assumption that (a^*, b^*) is a Nash equilibrium. An analogous argument applies when the reverse inequality holds, using the deviation $b = a^* - \varepsilon$. This completes the proof. □

C Proof of Proposition 3

Fix $c > 0$ and define, for $x \in [0, 1]$,

$$g(x) \;:=\; F\big(\min\{1, x + c\}\big) - 2F(x) + F\big(\max\{0, x - c\}\big).$$

Since F is continuous on $[0, 1]$, the function g is continuous on $[0, 1]$.

We evaluate g at the endpoints.

At $x = 0$, we have $\max\{0, 0 - c\} = 0$ and $\min\{1, 0 + c\} = \min\{c, 1\}$, hence

$$g(0) = F(\min\{c, 1\}) - F(0) \;\geq\; 0,$$

since F is nondecreasing.

At $x = 1$, we have $\min\{1, 1+c\} = 1$ and $\max\{0, 1-c\} = \max\{0, 1-c\}$, hence

$$g(1) = F(1) - 2F(1) + F(\max\{0, 1-c\}) = F(\max\{0, 1-c\}) - F(1) \ \leq \ 0.$$

Thus $g(0) \geq 0$ and $g(1) \leq 0$. By the Intermediate Value Theorem, there exists some $x \in [0, 1]$ such that $g(x) = 0$, i.e.

$$F(x+c) - F(x) = F(x) - F(x-c),$$

with the understanding that $F(x+c) = F(1)$ when $x+c > 1$ and $F(x-c) = F(0)$ when $x - c < 0$.

Therefore a conditional median exists. □

D Proof of Theorem 2

Let (a^*, b^*) be a Nash equilibrium of $G(F, c)$. Since the game is symmetric and zero-sum, equilibrium payoffs satisfy

$$P_A(a^*, b^*) = P_B(a^*, b^*) = 0.$$

From Theorem 1, a^* and b^* are conditional medians.

Suppose, toward a contradiction, that a^* is not window dominant.

Failure of global dominance. If a^* is not globally c-dominant, then for $I^* = [a^* - c, a^* + c]$ there exists an interval $J \subset [0, 1]$ of length $2c$ with $J \cap I^* = \varnothing$ such that

$$\rho(J) > \rho(I^*).$$

Let x^* be the midpoint of J. If player B deviates to x^*, then

$$P_B(a^*, x^*) = \big[\rho(J) - \rho(I^*)\big] \cdot 2c > 0,$$

contradicting the assumption that (a^*, b^*) is a Nash equilibrium.

Failure of local dominance. Alternatively, suppose a^* is not locally c-dominant. Then there exists $\delta \in (0, 2c]$ such that either

$$\rho([a^* + c, a^* + c + \delta]) > \rho([a^*, a^* + \tfrac{\delta}{2}]),$$

or

$$\rho([a^* - c - \delta, a^* - c]) > \rho([a^* - \tfrac{\delta}{2}, a^*]).$$

Without loss of generality, assume the first inequality holds.

Consider a deviation by player B to $b = a^* + \delta$. The indifference point is

$$m = \tfrac{a^* + (a^* + \delta)}{2} = a^* + \tfrac{\delta}{2}.$$

Player B's payoff is therefore

$$P_B(a^*, a^* + \delta) = \big[F(a^* + c + \delta) - F(a^* + \tfrac{\delta}{2})\big] - \big[F(a^* + \tfrac{\delta}{2}) - F(a^* - c)\big].$$

Since a^* is a conditional median,

$$F(a^*) - F(a^* - c) = F(a^* + c) - F(a^*).$$

Adding and subtracting these terms yields

$$\begin{aligned} P_B(a^*, a^* + \delta) &= -2\big[F(a^* + \tfrac{\delta}{2}) - F(a^*)\big] + \big[F(a^* + c + \delta) - F(a^* + c)\big] \\ &= \big[\rho([a^* + c, a^* + c + \delta]) - \rho([a^*, a^* + \tfrac{\delta}{2}])\big] \cdot \delta > 0, \end{aligned}$$

contradicting equilibrium. An analogous argument applies when the second inequality holds, using the deviation $b = a^* - \delta$. This establishes the necessity of window dominance.

Sufficiency. Now suppose a^* is a conditional median and is c-window dominant. We show that (a^*, a^*) is a Nash equilibrium.

At (a^*, a^*),

$$P_A(a^*, a^*) = P_B(a^*, a^*) = 0.$$

By symmetry, it suffices to show that for all $x \in [0, 1]$,

$$P_A(x, a^*) \leq 0.$$

Let $I^* = [a^* - c, a^* + c]$ and $J = [x - c, x + c]$. If $J \cap I^* = \varnothing$, then

$$P_A(x, a^*) = \big[\rho(J) - \rho(I^*)\big] \cdot 2c \leq 0$$

by global c-dominance.

If $J \cap I^* \neq \varnothing$, then $x = a^* \pm \delta$ for some $\delta \in (0, 2c]$. Consider $x = a^* + \delta$. The indifference point is $m = a^* + \frac{\delta}{2}$, and

$$P_A(x, a^*) = \big[F(a^* + c + \delta) - F(a^* + \tfrac{\delta}{2})\big] - \big[F(a^* + \tfrac{\delta}{2}) - F(a^* - c)\big].$$

Using the conditional median identity and rearranging as before,

$$P_A(x, a^*) = \big[\rho([a^* + c, a^* + c + \delta]) - \rho([a^*, a^* + \tfrac{\delta}{2}])\big] \cdot \delta \leq 0$$

by local c-dominance. The case $x = a^* - \delta$ is symmetric.

Thus $P_A(x, a^*) \leq 0$ for all $x \in [0, 1]$, and (a^*, a^*) is a Nash equilibrium. □

E Proof of Proposition 4

A unimodal symmetric distribution is a distribution with a density f such that $f(x)$ is non-decreasing in $[0, a]$, where a is the mode,

$$y > x \Rightarrow f(y) \geq f(x), \qquad x, y \in [0, a]$$

$f(x)$ is non-increasing in $[a, 1]$,

$$y > x \Rightarrow f(y) \leq f(x), \qquad x, y \in [a, 1]$$

and f is symmetric around the mode a.

$$f(a - x) = f(a + x) \qquad x \in [0, min(a, 1 - a)]$$

We illustrate global c-window dominance at a. For $I^* = [a - c, a + c]$, consider an interval J of length $2c$ centred around x such that $x < a$ and $J \cap I^* = \varnothing$. Then, the following inequalities hold

$$\rho(J) \leq \rho([x, x + c]) \leq \rho([a - c, a]) = \rho([a - c, a + c]) = \rho(I^*)$$

An analogous argument can be used for $x > a$ using a comparison with $\rho([a, a + c])$.
We now show local c-window dominance at a.

$$\rho([a + c, a + c + \delta]) \leq \rho([a + c, a + c + \frac{\delta}{2}]) \leq \rho([a, a + \frac{\delta}{2}])$$

and

$$\rho([a - c - \delta, a - c]) \leq \rho([a - c - \frac{\delta}{2}, a - c]) \leq \rho([a - \frac{\delta}{2}, a])$$

This completes the proof. □

F Proof of Proposition 5

A non-decreasing density f is characterised by the following condition:

$$y > x \Rightarrow f(y) \geq f(x) \qquad x, y \in [0, 1]$$

First, we show that there exists a conditional median $a \in [1 - c, 1]$, then show that a is c-window dominant.
Recall Proposition 2. Define the function $G(x)$ on $[0, 1]$ as

$$G(x) = [F(x + c) - F(x)] - [F(x) - F(x - c)]$$

Since F is a continuous function on $[0, 1]$, G is continuous on $[0, 1]$.
We evaluate G at $x = 1 - c$ and at $x = 1$. At $x = 1 - c$,

$$G(1 - c) = [F(1) - F(1 - c)] - [F(1 - c) - F(1 - 2c)] \geq 0$$

This is due to the non-decreasing nature of the density f. Now, at $x = 1$,

$$\begin{aligned} G(1) &= [F(1 + c) - F(1)] - [F(1) - F(1 - c)] \\ &= F(1 - c) - 1 \leq 0 \end{aligned}$$

Thus, $G(1-c) \geq 0$ and $G(1) \leq 0$. By the Intermediate Value Theorem, there exists $a \in [1-c, 1]$ such that a is a conditional median.
We go on to show that a is c-window dominant.
First, we illustrate global c-window dominance at a.
For $I^* = [a-c, a+c]$, consider an interval J of length $2c$ centred at $x \in [0,1]$ such that $J \cap I^* = \varnothing$. Then, the following inequalities hold:

$$\rho(J) \leq \rho([x, x+c]) \leq \rho([a-c, a]) = \rho([a-c, 1]) = \rho(I^*)$$

These inequalities hold since $J \cap I^* = \varnothing$ only for $x < a$. We now show local c-window dominance at a.

$$\rho\left([a-c-\delta, a-c]\right) \leq \rho\left(\left[a-c-\frac{\delta}{2}, a-c\right]\right) \leq \rho\left(\left[a-\frac{\delta}{2}, a\right]\right)$$

and

$$0 = \rho\left([a+c, a+c+\delta]\right) \leq \rho\left(\left[a, a+\frac{\delta}{2}\right]\right)$$

Thus, completing the proof for non-decreasing densities. We can use analogous arguments for non-increasing densities by showing the existence of a conditional median a on $[0, c]$ and then showing c-window dominance of a.

□